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Introduction

Fixed point theory of course entails the search for a combination of conditions on a set S and a mapping $T : S \to S$ which, in turn, assures that T leaves at least one point of S fixed, i.e. $x = T(x)$ for some $x \in S$. The theory has several rather well-defined (yet overlapping) branches. The purely topological theory as well as those topics which lie on the borderline of topology and functional analysis (e.g., those related to Leray-Schauder theory) have their roots in the celebrated theorem of L. E. J. Brouwer. This book is concerned with another branch of functional analytic theory, a branch which might be more properly viewed as a far reaching outgrowth of the contraction mapping principle of Banach and Picard. The mappings involved are of the form $T : K \to K$, K a subset of a Banach space, where T is *nonexpansive*; thus $\|T(x) - T(y)\| \le \|x - y\|$ for all $x, y \in K$.

Because of its important linkages with the theory of monotone and accretive operators, fixed point theory for nonexpansive mappings has long been considered a fundamental part of nonlinear functional analysis. However, the attempt to classify those subsets of Banach spaces which have the fixed point property for such mappings has now become a study in its own right – one which has yielded many elegant results and led to numerous discoveries in Banach space geometry. The aim of this book is to give a unified account of the major new developments inspired by B. Maurey's application in 1980 of Banach space ultraproducts to the theory.

To place these results in perspective we review some history. Fixed point theory for nonexpansive mappings has its origins in the 1965 existence theorems of F. Browder, D. Gohde, and W. A. Kirk. Although such mappings are natural extensions of the contraction mappings, it was clear from the outset that the study of nonexpansive mappings required techniques which go far beyond the purely metric approach. The traditional methods used in studying nonexpansive mappings have involved an intertwining of geometrical and topological arguments. Over the past twenty-five years these methods have yielded substantial results, both of a constructive and nonconstructive nature (the former not requiring the Axiom of Choice). For example, the original theorems of Browder and Gohde exploited the special geometrical structure of uniformly convex Banach spaces. The somewhat more general theorem of Kirk asserts that K has the fixed point property for nonexpansive mappings if K is weakly compact and convex and at the same time possesses a property called 'normal structure' (which means that any convex subset H of K which contains more than one point must contain a point z which has the property:

$$\sup\{\|z-y\| : y \in H\} < \sup\{\|x-y\| : x, y \in H\}).$$

Thus the underlying domain is assumed to possess both topological and geometric properties. At the same time it should be noted that while K is assumed to be compact in the weak topology, T is only norm continuous – thus the Schauder-Tychonoff theorem does not apply. (We also remark that while this theorem was initially established via an application of Zorn's lemma, constructive proofs are now known to exist.)

The early phases of the development of the theory centered around the identification of classes of spaces whose bounded convex sets possess normal structure, and it was soon discovered that certain weakenings and variants of normal structure also suffice. In terms of the classical spaces of functional analysis this approach yielded the following facts: All bounded closed and convex subsets of the ℓ^p and L^p spaces, $1 < p < \infty$, have the fixed point property for nonexpansive mappings, as do all weak* compact convex subsets of ℓ^1 (L. Karlovitz) and all order intervals (closed balls) in ℓ^∞ and L^∞ (R. Sine; P. Soardi). It was shown that the closed convex hull of a weakly convergent sequence in c_0 has this property also (E. Odell and Y. Sternfeld), and that bounded closed convex subsets of certain renormings of ℓ^2 which fail to have normal structure may still have the fixed point property for nonexpansive mappings (L. Karlovitz; J. B. Baillon and R. Schoneberg). A number of more abstract results were also discovered, along with important discoveries related both to the structure of the fixed point sets and to techniques for approximating fixed points. However, in concrete terms, the classical results described above represented the only substantial contributions to the existence part of the theory until 1980 when D. Alspach discovered an example (published in 1981) of a weakly compact convex subset of L^1 which fails to have the fixed point property for nonexpansive mappings. Alspach's example showed that *some* assumption in addition to weak compactness is needed and at the same time it set the stage for Maurey's surprising discovery that all bounded closed convex subsets of *reflexive subspaces* of L^1 do have the fixed point property for nonexpansive mappings. Maurey also showed that the same is true of all weakly compact convex subsets of c_0 and of the Hardy space H^1.

Maurey's methods are nonconstructive in the sense that they involve the Banach space ultraproduct construction. Ultraproduct methods are fundamental to set theory and over the years have found applications in many other branches of mathematics, including both algebra and analysis. However the set-theoretic ultraproduct of Banach spaces in general is not a Banach space and the modification used by Maurey was not put in explicit form until 1972 when it was introduced by D. Dacunha-Castelle and J. Kirvine. Since 1980 Banach space ultraproducts and related ultraproduct methods have yielded many results in fixed point theory for nonexpansive mappings. Also, in those instances where 'standard' techniques have subsequently been shown to yield the same results, the standard approach often appears to be neither simpler nor more intuitive.

Because the methods involved are fairly sophisticated, roughly the first third of this book is devoted to laying a careful foundation for the actual fixed point-theoretic results which follow. The text begins with a careful review of the concepts of Schauder bases, filters, ultrafilters, limits over ultrafilters, and nets. In Chapter 2 both the set-theoretic and Banach space ultraproduct constructions are presented in detail, and finite representability, the Banach-Saks properties, and the ultraproduct of mappings are also discussed. The chapter concludes with a description of the fundamental spaces of Tzirelson and James. The final chapter begins with an introduction to the classical approach to the theory, including a discussion of normal structure. Then, after translating these basic results into ultraproduct language, several new fixed point theorems, including the results of Maurey, are presented. The chapter concludes with a recent application of ultranets due to Kirk.

A major part of the classical fixed point theory for nonexpansive mappings may be found in the recent book: *Topics in Metric Fixed Point Theory* by K. Goebel and W. A. Kirk. However, the nonstandard methods are treated only briefly by Goebel and Kirk; thus the present work might be viewed as complementary to that book. At the same time it should be emphasized that the treatment given here is self-contained in the sense that all results pertinent to the development are included.

—W. A. Kirk

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