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Homogenization of Reticulated Structures

With 110 Illustrations

Springer

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To Jacques-Louis Lions

Preface

This book presents recent works on lattice type structure. Some of the results discussed here have already been published in mathematical journals, but we give here a comprehensive and unified presentation. We have also added some new topics such as those contained in Chapter 4 treating elastic problems for gridworks. The aim of this book is to give continuous simple models for thin reticulated structures (which may have a very complex pattern). This means that we have to treat partial differential equations depending on several small parameters and give the asymptotic behavior with respect to these parameters (which can be the period, the thickness of the material, or the thickness of a plate or of a beam).

This book is written from the point of view of the applied mathematician, attention being paid to the mathematical rigor, convergence results, and error estimates. It consists of six chapters and more than a hundred figures. The basic ideas are presented in the first two chapters, while the four last ones study some particular models, using the ideas of Chapters 1 and 2.

Chapter 1 is an introduction to homogenization methods in perforated domains. Here the parameter to be taken into consideration is the period. After describing the multiple-scale method (which consists in asymptotic expansions), we focus our attention on the variational method introduced by Tartar, whose main idea is the construction of rapidly oscillating test functions.

Chapter 2 gives the main ideas for the study of thin reticulated structures. In this step, the relevant parameter is the thickness of the bars or layers that form the structure. The method consists in a change of scale in order to work on a fixed domain.

Chapters 3 and 4 are dedicated to the study of networks in thermal and elasticity problems. This means that we have to take into account an additional parameter, which is the thickness of the network considered as a plate. So we start with a three-dimensional system and end up with a two-dimensional one.

Chapters 5 and 6, which are independent of Chapters 3 and 4, treat the same kind of problems, but now the structure is a thin, tall one (like a tower), and the

additional small parameter is now the thickness of the structure considered as a beam. We again start with a three-dimensional system, but here we end up with a one-dimensional problem.

The close friendship and long collaboration between the authors originated when Professor Jacques-Louis Lions introduced them to the subject of homogenization and the asymptotic study of reticulated structures. The authors take here the opportunity to warmly thank him, and to dedicate this book to him, as an expression of their deep gratitude.

We also address our thanks to Professor Klaus Kirchgässner, who gave us the idea of writing this book.

We thank Sylvie Maggipinto for her careful typing of the manuscript.

Finally, we are very grateful to our colleague Thomas Lachand-Robert who is a $\text{\TeX}$ wizard and whose help was invaluable.

Doina Cioranescu
Jeannine Saint Jean Paulin

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Prerequisites and Notation

In each chapter of the book, theorems, corollaries, remarks, and examples are numbered consecutively, the first number indicating the section. Figures are numbered consecutively from the beginning to the end of the book. As usual, $\square$ indicates the end of the proofs.

Throughout the book the Einstein summation convention on repeated indices is adopted.

The essential mathematical prerequisites to read this book are the following topics: elements of functional analysis such as distributions; L^p and Sobolev spaces and related properties; weak and weak $\star$ convergences; elementary results in the study of partial differential equations of elliptic type, as for instance, the Lax–Milgram theorem; regularity of solutions; continuous dependence on the data.

Concerning these topics, we refer the reader to Adams (1975), Brezis (1987), Gilbarg and Trudinger (1997), Lions and Magenes (1972), Rudin (1973, 1974).

The following notation and properties (which all can be found in the above references) will be used in the sequel:

ε , respectively δ : The general term of a sequence of positive numbers which converges to zero.

C : Different constants independent of small parameters taken into consideration.

$\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$: Respectively, the set of real numbers, the set of natural numbers, the set of integers, and the set of rational numbers.

δ_{ij} : The Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Ω : A bounded open set in $\mathbb{R}^n$.

$\partial\Omega$: The boundary of Ω .

$\text{meas } \Omega$: The Lebesgue measure of Ω (n -dimensional if Ω is an n -dimensional open set, $(n - 1)$ -dimensional if Ω is a curve).

n : The unit exterior normal vector with respect to Ω .

$\mathcal{L}(V; W)$: The space of linear and continuous applications from V to W , where V and W are real Banach spaces.

V' : The dual space of the real Banach space V , i.e. $V' = \mathcal{L}(V, \mathbb{R})$.

$\langle \cdot, \cdot \rangle_{V', V}$: The duality pairing between V' and V .

$a(\cdot, \cdot)$: A bilinear continuous form from $V \times V$ to $\mathbb{R}$, i.e., linear with respect to each variable and satisfying

$$|a(u, v)| \leq C \|u\|_V \|v\|_V \quad \forall u, v \in V,$$

where C is a constant independent of u and v .

$C^\infty(\mathbb{R}^n)$: The space of indefinitely differentiable functions defined on $\mathbb{R}^n$.

$\mathcal{D}(\Omega)$: The space of indefinitely differentiable functions with compact support in Ω .

$\mathcal{D}'(\Omega)$: The space of distributions on Ω .

$L^p(\Omega)$: The space of measurable functions such that

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \int_{\Omega} |f(x)|^p dx < +\infty \quad \text{if } 1 < p < \infty, \\ \|f\|_{L^\infty(\Omega)} &= \inf\{C, |f| \leq C, \text{ a.e. in } \Omega\} \quad \text{if } p = \infty. \end{aligned}$$

$W^{1,p}(\Omega)$: The Sobolev space defined for $1 < p < \infty$ by

$$W^{1,p}(\Omega) = \left\{ f \mid f \in L^p(\Omega), \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, \dots, n \right\},$$

which is a Banach space for the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left(\|f\|_{L^p(\Omega)}^p + \|\text{grad } f\|_{L^p(\Omega)}^p \right)^{1/p},$$

where

$$\|\text{grad } f\|_{L^p(\Omega)} = \left(\sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{1/p}$$

(with an obvious modification for $p = \infty$).

$W^{-1,p'}(\Omega)$: The dual space of $W^{1,p}(\Omega)$, where

$$\frac{1}{p'} = 1 - \frac{1}{p}.$$

$W_0^{1,p}(\Omega)$: The closure of $\mathcal{D}(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$.

$H^1(\Omega)$: The space $W^{1,2}(\Omega)$, which is a Hilbert space for the scalar product

$$(f, g)_{H^1(\Omega)} = (f, g)_{L^2(\Omega)} + \left(\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right)_{L^2(\Omega)}, \quad \forall f, g \in H^1(\Omega).$$

$\gamma(f)$: The trace of a function $f \in H^1(\Omega)$.

$H^{\frac{1}{2}}(\partial\Omega)$: The space of traces of the functions in $H^1(\Omega)$ for $\partial\Omega$ sufficiently smooth. It is a Banach space for the norm

$$\|f\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 = \int_{\partial\Omega} f^2 ds + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+1}} ds ds',$$

and there exists a constant C depending on Ω such that

$$\|\gamma(f)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega).$$

$H_0^1(\Omega)$: The space $W_0^{1,2}(\Omega)$, which can also be characterized as follows:

$$H_0^1(\Omega) = \{f \mid f \in H^1(\Omega), \gamma(f) = 0\}.$$

$H^{-1}(\Omega)$: The dual space of $H_0^1(\Omega)$.

Y : A reference cell in $\mathbb{R}^n$, i.e.,

$$Y = (0, \ell_1) \times \cdots \times (0, \ell_n),$$

where $\ell_1, \dots, \ell_n$ are positive numbers. A function f , defined a.e. on $\mathbb{R}^n$, is called Y -periodic if

$$f(x_1 + k_1 \ell_1, \dots, x_n + k_n \ell_n) = f(x) \quad \text{a.e. on } \mathbb{R}^n,$$

for any $k_i \in \mathbb{Z}$, $i \in \{1, \dots, n\}$.

$C_{\text{per}}^\infty(Y)$: The subset of $C^\infty(\mathbb{R}^n)$ of Y -periodic functions.

$H_{\text{per}}^1(Y)$: The closure of $C_{\text{per}}^\infty(Y)$ for the H^1 -norm. If $f \in H_{\text{per}}^1(Y)$, then the trace $\gamma(f)$ on the opposite faces of Y has the same value. This is the sense to be given to the notion of periodicity for functions in $H^1(Y)$.

χ_Ω : The characteristic function of the set $\Omega \subset \mathbb{R}^n$, i.e.,

$$\chi_\Omega(x) = \begin{cases} 1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

$\tilde{v}$: The extension by zero on Ω of any function v defined on a subset ω of Ω , i.e.,

$$\tilde{v} = \begin{cases} v & \text{for } x \in \omega, \\ 0 & \text{for } x \in \Omega \setminus \omega. \end{cases}$$

$\mathcal{M}_\Omega(\phi)$: The mean value (or the average) of a function $\phi \in L^1(\Omega)$ over Ω , i.e.,

$$\mathcal{M}_\Omega(\phi) = \frac{1}{\text{meas } \Omega} \int_\Omega \phi(x) dx.$$

The following essential results (for details, see Adams (1975), Brezis (1987), Lions and Magenes (1972), Rudin (1973, 1974)) will be used frequently:

Average Convergence for Periodic Functions. Let Ω be a bounded open set in $\mathbb{R}^n$. Let f be a Y -periodic function such that $f \in L^p(Y)$ with $1 \leq p \leq +\infty$. Define

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. for } x \in \mathbb{R}^n.$$

Then, as $\varepsilon \rightarrow 0$,

$$f_\varepsilon \rightharpoonup \mathcal{M}_Y(f) \quad \text{weakly in } L^p(\Omega)$$

if $p < +\infty$. The convergence is weak $\star$ in $L^\infty(\mathbb{R}^n)$, if $p = +\infty$.

Sobolev Embeddings Theorem. If $\partial\Omega$ is sufficiently smooth, one has the following embeddings:

(i) If $1 \leq p < n$, $W^{1,p}(\Omega) \subset L^q(\Omega)$ with

$$\text{compact injection for } q \in [1, n^*[, \text{ where } \frac{1}{n^*} = \frac{1}{p} - \frac{1}{n},$$

continuous injection for $q = n^*$.

(ii) If $p = n$, $W^{1,n}(\Omega) \subset L^q(\Omega)$, with compact injection if $q \in [1, +\infty[$.

(iii) If $p > n$, $W^{1,p}(\Omega) \subset C^0(\Omega)$ with compact injection.

(iv) The injection $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ is compact.

(v) The injection $L^2(\Omega) \subset H^{-1}(\Omega)$ is compact.

Extension Theorem. If $\partial\Omega$ is sufficiently smooth, there exists an extension operator $P \in \mathcal{L}(H^1(\Omega); H^1(\mathbb{R}^n))$ such that

$$(i). \quad Pf = f \quad \text{on } \Omega;$$

$$(ii). \quad \|Pf\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)},$$

$$(iii). \quad \|Pf\|_{H^1(\mathbb{R}^n)} \leq C\|f\|_{H^1(\Omega)},$$

where the constant C depends on Ω .

Remark. We supposed here that $\partial\Omega$ is sufficiently smooth. To be more precise, all the above theorems are valid if $\partial\Omega$ is Lipschitz continuous. Smooth domains are in particular, open balls in $\mathbb{R}^n$, polygons in $\mathbb{R}^2$, or polyhedra in $\mathbb{R}^3$.

Lax–Milgram Theorem. Let a be a bilinear continuous form on a Hilbert space H and let $f \in H'$ be given. Suppose that a is elliptic, i.e., there exists a positive constant α such that

$$a(v, v) \geq \alpha \|v\|_H^2.$$

Then the following problem has a unique solution $u \in H$:

$$\text{Find } u \in H \text{ such that } a(u, v) = \langle f, v \rangle_{H', H}, \quad \forall v \in H.$$

Moreover, u satisfies the estimate

$$\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'}.$$

Introduction

Composite materials are very widely used nowadays in civil engineering, electrotechnics, and the aero-space industry. A well-known example among thousands of others is a superconducting multifilamentary composite (SMC). Its technological interest lies in its ability to transport very large densities of electrical current under the influence of a high magnetic field. For physical reasons, a superconducting material is divided into elementary fibers with a very small diameter (from 10 to 100 μm). They are isolated from each other by a matrix with high thermal conductivity. There are up to several thousands of fibers. This is why such a material is called SMC.

In fluid mechanics, we have, for instance, suspensions of elastic or rigid particles or fluid flows in porous media. Such materials are considered in the oil-recovery industry.

In solid mechanics, other examples are composites, such as reinforced concrete or directional reinforced composites or foliated materials (Figures 1 and 2).

Figure 1. A composite material.

Figure 2. A foliated material.

The above examples are materials with a large number of heterogeneities (inclusions or holes). When trying to characterize the properties of such a material,

one can do it on a *local scale*. This means that one takes into consideration each component of the composite separately. But in practice, it is much more interesting to characterize the composite on a *global scale*. It is more important to give an overall (or effective) behavior of the composite by neglecting the fluctuations due to the heterogeneities.

This is, in fact, the aim of *homogenization theory*. The heterogeneous material is replaced by a fictitious one—the homogenized material—whose behavior must be as close as possible to the behavior of the composite itself. When doing so, one tries to describe the global properties of the composite by taking into account the local properties of its constituents.

Let us point out that in general, a good global description is not obtained merely by averaging the local characteristics of the composite. Indeed, for composite materials, the homogenized coefficients are decomposed into two parts: one is the usual average of physical coefficients (mixture law), while the other one corrects the mixture law and is of the same order as the first part. For the periodic case, the homogenization process consists in replacing the initial partial differential equation with rapidly oscillating coefficients that describe the composite material by a partial differential equation with constant coefficients that characterize the fictitious homogenized one. These coefficients (the homogenized coefficients) are obtained by solving a nonoscillating partial differential equation on a period of reference.

At this point, let us also mention the numerical interest of homogenization methods. If the inclusions are small and numerous (in SMC, there may be thousands of fibers), a direct numerical computation of the solution becomes difficult, because it needs a very fine discretization mesh, which means a significant computation time, and such a computation is often irrelevant. As we mentioned above, for the homogenized solution, we have to solve problems in which no rapid oscillation occurs, so no numerical difficulty appears. Moreover, error estimates can be obtained in most cases, showing that when the period is small enough, the homogenized solution is a good approximation to the physical one.

More complex yet are the *perforated materials*, where the holes play the role of inclusions. The term “perforated” means that the state equation is written in the matrix solely and that there are boundary conditions on the whole boundary of the matrix, the boundary of the holes included.

From a mathematical point of view, one can consider that we have “holes” in the regions where no equation is written down. The mathematical holes may be physical ones. They may also correspond to a material other than that of the matrix. For example, in the case of superconducting multifilamentary composites in a weak magnetic field, the equations are written down only in the conducting matrix.

Similarly, in a porous medium, Stokes or Navier-Stokes equations are written only in the fluid part (the rocks playing the role of “mathematical” holes). Though the coefficients are in general constant, when we are in the presence of many small holes, the same mathematical and numerical difficulties as in the case of composites appear, due to the great number of interface matrix-holes. As we will

see in Chapter 1, homogenization methods can be applied to perforated materials, and they give good overall approximations.

The aim of our study here is to give the global behavior of a class of complicated perforated media: periodic structures with big holes and a very small amount of material concentrated along layers (*honeycomb* structures) or along bars (*reinforced* structures) as shown in Figures 3 and 4.

In these cases, we have periodicity in three directions. Besides the period, another small parameter plays a role in the description of the global behavior; it is precisely the small quantity of material.

Figure 3. Honeycomb structure.

Figure 4. Reinforced structure.

Still more complex structures are *grids*, *towers* (tall honeycomb structures) and *cranes* (tall reinforced structures). In these structures, we have periodicity only in two or in one directions (Figures 5–7). Here a new small parameter occurs; it is the small thickness (in one or two directions) of these structures: the thickness of the grid considered as a perforated plate or the thickness of the towers and cranes considered as perforated beams.

Figure 5. A grid.

Figure 6. A tall honeycomb structure.

Figure 7. A tall reinforced structure.

In the mathematical study we perform here, we take into consideration these three small parameters, i.e., the period, the small quantity of material, and the thickness. When studying the dependence on the period, we use homogenization methods (such as the multiple-scale method and variational methods) in the directions in which we have periodicity. For the dependence on the small amount of material, we give a perturbation method that enables us to give explicitly a good approximation of the real structure. The dependence on the third small parameter is studied by plate and beam techniques. We also give error estimates and some numerical results.

It is noteworthy that in all the cases we treat, we end up with simple partial differential equations with constant coefficients that are explicit algebraic expressions of the “physical” ones.