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Complex Tori

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Introduction

A *complex torus* is a connected compact complex Lie group. Any complex torus is of the form $X = \mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$. A meromorphic function on $\mathbb{C}^g$, periodic with respect to Λ, may be considered as a function on X. An *abelian variety* is a complex torus admitting sufficiently many meromorphic functions. In other words, abelian varieties are exactly the algebraic complex tori. Thus abelian varieties are special complex tori. In fact, a general complex torus of dimension ≥ 2 does not admit any meromorphic function. Whereas abelian varieties are very well investigated – they have been studied for almost two hundred years now – not much is known about arbitrary complex tori, although they are among the simplest complex manifolds. There are very few papers and no book dealing with nonalgebraic complex tori. On the other hand, as the following examples show, these tori frequently occur even if one starts within the category of abelian varieties:

(i) Griffiths intermediate Jacobian of an abelian variety of dimension ≥ 3 is a complex torus, which in general is not an abelian variety (see Section 4.2).

(ii) Let X_1 and X_2 be abelian varieties. There are uncountably many extensions $0 \to X_1 \to X \to X_2 \to 0$ as complex tori. Among them there are only countably many for which X is an abelian variety (see Section 1.6). Hence a general extension of abelian varieties is not an abelian variety.

Apart from their importance for applications, for example in the theory of algebraic cycles via intermediate Jacobians, not necessarily algebraic complex tori are interesting for their own sake, and this is the subject of the present book. The central topics are those properties of complex tori which are actually different from the corresponding properties of abelian varieties. The most important such topics are:

(1) An important feature of abelian varieties is that any complex subtorus admits a complement up to isogeny. This leads to Poincaré's Reducibility Theorem. For complex tori this is not valid any more. It is easy to construct complex tori admitting only one nontrivial complex subtorus, so one has to distinguish between simple and indecomposable complex tori (see Section 1.7).

(2) It is a consequence of Poincaré's Reducibility Theorem that the endomorphism algebra $\text{End}_{\mathbb{Q}}(X)$ of an abelian variety X is semisimple. This is not the case for an arbitrary complex torus X. We will show, however, that the semisimplification $\text{End}_{\mathbb{Q}}(X)_{ss}$ of $\text{End}_{\mathbb{Q}}(X)$ decomposes in the same way as $\text{End}_{\mathbb{Q}}(X)$ for abelian varieties (see Section 1.8).

(3) The positivity of the trace form yields strong restrictions for a finite dimensional $\mathbb{Q}$-algebra to occur as the endomorphism algebra of an abelian variety. The theorem of Oort-Zarhin says that on the other hand, for any finite dimensional $\mathbb{Q}$-algebra A there is a complex torus X with $\text{End}_{\mathbb{Q}}(X) = A$. We present a proof, due to Hötte-Scharlau, in Section 1.9.

(4) A *polarization* of an abelian variety $X = \mathbb{C}^g/\Lambda$ is a positive definite hermitian form H on $\mathbb{C}^g$, whose imaginary part is integer valued on the lattice Λ. A complex torus which is not an abelian variety does not, by definition, admit a polarization. But there is a replacement for this: The index of a nondegenerate hermitian form is the number of negative eigenvalues. If, instead of being positive definite, the hermitian form H is nondegenerate of index k, we call H a *polarization of index* k. The pair (X, H) is called a *nondegenerate complex torus of index* k. In these terms polarized abelian varieties are nondegenerate complex tori of index 0. The investigation of nondegenerate complex tori is the subject of Chapter 2.

(5) The restriction of a polarization of an abelian variety to a complex subtorus is also a polarization. This is not valid for polarizations of positive index. It is shown in Section 2.5, however, that if (X, H) is a nondegenerate complex torus of index k, and Y is a complex subtorus such that $H|Y$ is nondegenerate, then Y admits a complementary complex subtorus Z. Moreover, $H|Z$ is nondegenerate.

This leads to a modification of Poincaré's Reducibility Theorem: Call a nondegenerate complex torus (X, H) *irreducible* if X does not admit any nontrivial subtorus Y, such that $H|Y$ is nondegenerate. Any nondegenerate complex torus is isogenous to a product of irreducible nondegenerate complex tori (see Section 2.5).

(6) Abelian varieties are exactly those complex tori which admit a holomorphic embedding into some projective space. Thus a complex torus X that is not an abelian variety does not admit a holomorphic projective embedding. However, if X is of dimension g and H a polarization of index k on X, then X admits a differentiable embedding into projective space which is holomorphic in $g - k$ variables and antiholomorphic in k variables (see Chapter 3). In fact, choose a line bundle L on X with first Chern class H. Then $H^k(X, L)$ is the only nonvanishing cohomology group of L. The vector space $H^k(X, L)$ may be considered as the vector space of harmonic forms of type $(0, k)$ with values in L. Choosing a suitable metric of L, these forms define a differentiable map $\varphi_L : X \dashrightarrow \mathbb{P}_N$ on an open dense set of X, holomorphic in $g-k$ variables and antiholomorphic in k variables. An analogue of a theorem of Lefschetz says that φ_{L^n} is an embedding for $n \geq 3$ (see Section 3.6).

(7) Let $(X = \mathbb{C}^g/\Lambda, H)$ be a nondegenerate complex torus of index k. If $d_1, \dots, d_g$ denote the elementary divisors of the alternating form $\operatorname{Im} H$ on Λ, the polarization H is called *of type* $(d_1, \dots, d_g)$. A *symplectic marking* of (X, H) is a basis of the lattice Λ symplectic with respect to $\operatorname{Im} H$. We show in Section 7.4 that there exists an analytic fine moduli space $\mathcal{H}_{g,k}$ for nondegenerate complex tori of index k and type $(d_1, \dots, d_g)$ with symplectic marking. Note that $\mathcal{H}_{g,0}$ and $\mathcal{H}_{g,g}$ are just the Siegel upper and lower half space. The group $Sp_{2g}(\mathbb{Z})$ acts on $\mathcal{H}_{g,k}$ in such a way that the quotient $\mathcal{H}_{g,k}/Sp_{2g}(\mathbb{Z})$ parametrizes nondegenerate complex tori of index k and type $(d_1, \dots, d_g)$ (see Section 2.2). Moreover, for $k = 0$ and g, the quotient $\mathcal{H}_{g,k}/Sp_{2g}(\mathbb{Z})$ is an analytic coarse moduli space for abelian varieties of type $(d_1, \dots, d_g)$. For $1 < k < g$, $\mathcal{H}_{g,k}/Sp_{2g}(\mathbb{Z})$ is only a topological coarse moduli space. The fact that $\mathcal{H}_{g,k}/Sp_{2g}(\mathbb{Z})$ does not admit any complex structure (see Section 7.2) implies that an analytic coarse moduli space for nondegenerate complex tori of index $k \neq 0, g$ and type $(d_1, \dots, d_g)$ does not exist (see Section 7.4).

(8) The endomorphism algebra of a simple complex torus X is a skew-field of finite dimension over $\mathbb{Q}$. As we mentioned above $\operatorname{End}_{\mathbb{Q}}(X)$ is completely arbitrary, whereas for an abelian variety there are

strong restrictions. However, the existence of a polarization H of index k on X gives some restrictions on $\mathrm{End}_{\mathbb{Q}}(X)$. Similarly as in the case of abelian varieties, the hermitian form H induces an anti-involution on $\mathrm{End}_{\mathbb{Q}}(X)$, the *Rosati-involution.* The skew-fields F of finite dimension over $\mathbb{Q}$ with anti-involution $'$ were classified by Albert. There are roughly three types (see Chapter 5):

Type Ia: F is an algebraic number field and $'$ is the identity.
Type Ib: F is a quaternion algebra over an algebraic number field and $'$ is conjugate to the canonical involution on F.
Type II: F is of degree d^2 over an algebraic number field K and $'$ restricts to an involution on K.

Shimura defined the notion of endomorphism structure of an abelian variety. He constructed, for any skew-field of finite dimension over $\mathbb{Q}$ with positive anti-involution, families of abelian varieties with corresponding endomorphism structure. Moreover he showed that every such abelian variety is contained in one of these families. The notion of endomorphism structure immediately generalizes to nondegenerate complex tori of index k. In Chapter 5 we construct, for any of the above mentioned skew-fields F with anti-involution $'$, families of nondegenerate complex tori with endomorphism structure in $(F, ')$ such that any such complex torus is contained in one of these families.

(9) Let $\mathcal{P}$ denote the parameter space of one of the families of nondegenerate complex tori just mentioned. In the case of abelian varieties there are three series of irreducible hermitian symmetric spaces of the noncompact types, namely C I, A III, and D III, such that any $\mathcal{P}$ is a product of members of these families. In the case of nondegenerate complex tori, $\mathcal{P}$ need not be a hermitian symmetric space anymore. Let G denote a real form of a connected complex semisimple Lie group $G_{\mathbb{C}}$. A *flag domain for* G is an open G-orbit in a complex flag manifold for $G_{\mathbb{C}}$. In Chapter 6 we show that there are ten series of flag domains for classical groups such that any parameter space is a product of their members. We shall see that, conversely, for every classical group except $GL_{2n+1}(\mathbb{R}), O_{2p+1,q}(\mathbb{R})$ and $O_{2n+1}(\mathbb{C})$ there is a flag domain parametrizing nondegenerate complex tori with endomorphism structure. In Section 7.5 we show that the spaces $\mathcal{P}$ are analytic fine moduli spaces for a suitable moduli problem.

(10) Let $\mathcal{P} = H\backslash G$ be one of the above parameter spaces of nondegenerate complex tori with endomorphism structure. There is an

arithmetic group Γ acting on $\mathcal{P}$ such that two points of $\mathcal{P}$ represent isomorphic nondegenerate complex tori with endomorphism structure if and only if they differ by an element of Γ. In order to see which of the quotient spaces $\mathcal{P}/\Gamma$ are coarse moduli spaces of the corresponding moduli problem, we have to distinguish two cases: If H is a compact group, $\mathcal{P}/\Gamma$ is an analytic coarse moduli space for the corresponding moduli functor. If H is noncompact, $\mathcal{P}/\Gamma$ is only a topological coarse moduli space. Moreover, in this case $\mathcal{P}/\Gamma$ does not admit the structure of a complex analytic space, so there does not exist an analytic coarse moduli space for the corresponding moduli functor (see Section 7.5).

As for prerequisites, we freely use the basic language of algebraic geometry and complex analysis as outlined for example in [GH]. The first three chapters of our book [CAV] deal with complex tori. In the first sections of Chapter 1 we present the results of these chapters that are needed subsequently without repeating proofs.

Most of the main results presented in this book seem to be new. We would, however, like to emphasize the strong influence of the papers [OZ1], [OZ2] of Oort and Zarhin and [Sh] of Shimura. Not new of course are the Theorem of Oort-Zarhin (see Section 1.8) for which we copied the elegant proof of Hötte-Scharlau (see [HS]) and Chapter 4 in which we present the intermediate Jacobians of Griffiths, Weil and Lazzeri as the most important examples for nondegenerate complex tori. Finally we included some well-known elementary results on complex tori and their moduli spaces.

Finally a few words about our cross-references. If we refer to Theorem 3.5 in Chapter 6, we write Theorem 6.3.5. If we are within Chapter 6, we omit the numbers of the chapter. Similarly, Section 7.2 means Section 2 of Chapter 7. Proposition A.6 stands for Proposition 6 of Appendix A.

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