Euclid—The Creation of Mathematics

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Euclid—The Creation of Mathematics

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Preface

This book is for all lovers of mathematics. It is an attempt to understand the nature of mathematics from the point of view of its most important early source.

Even if the material covered by Euclid may be considered elementary for the most part, the way in which he presents it has set the standard for more than two thousand years. Knowing Euclid's *Elements* may be of the same importance for a mathematician today as knowing Greek architecture is for an architect. Clearly, no contemporary architect will construct a Doric temple, let alone organize a construction site in the way the ancients did. But for the training of an architect's aesthetic judgment, a knowledge of the Greek heritage is indispensable. I agree with Peter Hilton when he says that genuine mathematics constitutes one of the finest expressions of the human spirit, and I may add that here as in so many other instances, we have learned that language of expression from the Greeks.

While presenting geometry and arithmetic Euclid teaches us essential features of mathematics in a much more general sense. He displays the axiomatic foundation of a mathematical theory and its conscious development towards the solution of a specific problem. We see how abstraction works and enforces the strictly deductive presentation of a theory. We learn what creative definitions are and

how a conceptual grasp leads to the classification of the relevant objects. Euclid creates the famous algorithm that bears his name for the solution of specific problems in arithmetic, and he shows us how to master the infinite in its various manifestations.

One of the greatest powers of scientific thinking is the ability to uncover truths that are visible only “to the eyes of the mind,” as Plato says, and to develop ways and means to handle them. This is what Euclid does in the case of irrational, or incommensurable, magnitudes. And finally, in the *Elements* we find so many pieces of *beautiful* mathematics that are easily accessible and can be studied in detail by anybody with a minimal training in mathematics.

Seeing such general phenomena of mathematical thinking that are as valid today as they were at the time of the ancient Greeks, we cannot but agree with the philosopher Immanuel Kant, who wrote in 1783 in the popular introduction to his philosophy under the heading “Is metaphysics possible after all?”:

There is no book at all in metaphysics such as we have in mathematics. If you want to know what mathematics is, just look at Euclid's *Elements*.

(Prolegomena §4 in free translation)

To be sure, some fundamental concepts needed for today's mathematics are absent from Euclid's *Elements*, most notably algebraic formulae and the concepts of a function and the real numbers. This does not, however, affect us when we study, for instance, how the exclusion of certain means of construction forces a more rigorous treatment of a piece of mathematics or how generalizations can be very profound on one occasion and shallow on another one.

In order to substantiate these general claims, we have to look at Euclid's *Elements* in detail. Our comments are based on the translation from the Greek original by Th. L. Heath (1908/1926). We will quote many of Euclid's definitions and theorems, but cite proofs only for a few of the most important results like Pythagoras's theorem or the irrationality of $\sqrt{2}$. Many of Euclid's other proofs are reproduced in their main outlines and in a mildly modernized language to be more easily accessible to the reader. For any questions about specific details the interested reader is advised to have Heath's or any other

translation into a modern language at hand. All recent translations are reliable, as my friends competent in classical philology tell me. If there are any problems in understanding a passage, they are almost always of a mathematical, and not a linguistic, nature and of minor importance. There are none of the ambiguities in a mathematical text as there are, for instance, in a novel or a philosophical treatise. However, in the *Elements* we still have a very distant historical text that needs interpretation. Here the present book takes a clear position: The *Elements* are read, interpreted, and commented upon from the point of view of modern mathematics.

I would like to share my joy and enthusiasm in studying Euclid with as many interested readers as possible. The *Elements* definitely deserve to be more widely known and read, and not just in the narrow circle of specialists in the history of mathematics. Nevertheless, I hope that the experts will find one or another aspect of my interpretation new and interesting.

Over the years, and during various stages of the preparation of this book, I have been helped and encouraged very generously by many people. Thanks to you all:

First and foremost, Maresia Artmann, then U. Artmann, L. Berggren, D. Fowler, C. Garner, J. Hainzl, R. Hersh, P. Hilton, K. H. Hofmann, W. Jonsson, V. Karasmanis, H. Knell, K. Lengnink, A. Mehl, Heike Müller, Ian Mueller, E. Neuenschwander, M. L. Niemann, S. Prediger, H. Puhlmann, L. Schäfer, M. Taisbak and his helping hand, S. Unguru, H.-J. Waschkies.

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David Hilbert about Euclid:

The sequence of our theorems will differ greatly from what one usually finds in textbooks on elementary geometry. It will, however, frequently be the same as in Euclid's *Elements*. Thus we will be led by our most modern investigations to appreciate the acumen of this ancient mathematician and to admire him in the highest degree.

(Die Reihenfolge unserer Theoreme wird von der in den Lehrbüchern der Elementargeometrie üblichen stark abweichen, sie wird dagegen vielfach übereinstimmen mit der Reihenfolge bei Euklid. So führen uns diese ganz modernen Untersuchungen dazu, den Scharfsinn dieses alten Mathematikers recht zu würdigen and aufs höchste zu bewundern.)

D. Hilbert: **Elemente der Euklidischen Geometrie.** Lecture Notes of a course in the winter term 1898/99, p. 44/45. In this course he prepared his famous *Grundlagen der Geometrie* (1899), but gave many more details than in the published version.

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Notes to the Reader

A large part of Euclid's *Elements* treats a body of mathematics that today would be called elementary geometry and arithmetic. Even today, textbooks for a high-school course in geometry, for example the popular H. R. Jacobs "Geometry" [1987], follow, for large parts, Euclid's *Elements* rather closely. One can understand, for instance, what is said about similarity in Euclid's Book VI without reading much of the preceding books. While it is advisable to follow Euclid step by step, it is not absolutely necessary. A reader with a solid background in high-school geometry can enter the discussion at any particular place to follow a particular interest. Chapter I, containing a short description of the contents the thirteen books of the *Elements*, is included for the convenience of those readers who are interested in special subjects. Hartshorne [2000] gives a radically modern presentation of Books I–VI.

In our main text, the description and comments on each of Euclid's books will be followed by more general remarks about typical mathematical procedures, subjects of particular historical interest, connections to philosophy, and similar items.

The 13 books (= chapters) of the *Elements* are denoted by Roman numerals, the propositions in each Book by Arabic numerals. Hence Prop. II.14 is Proposition 14 of Book II. Definitions are indicated by Def. II.2.

Propositions and definitions are quoted from the translation of Euclid by T. L. Heath [1926] or, in a sometimes slightly modernized version, Mueller [1981]. As mentioned in the Preface, I have tried to explain the mathematical content of the *Elements* to readers with a general interest in mathematics and its history. For more specialized questions one should consult Heath's, Vitrac's or any other translation. Aside from presenting the complete text, Heath discusses a great wealth of historical and other related details. He points out occasional little gaps in Euclid's arguments, and lists alternative proofs and similar items. He does not, however, speak about the general mathematical outlines of the text; he refrains from any judgment of the relative importance of the various propositions or theories and does not call to the attention of his readers general features of mathematics. All of these are of special interest for the present book.

Notes for the text are collected at the end of the book. There are very few numbered footnotes in the traditional sense. In the notes relevant books and papers are listed together with a few supplementary remarks. Whenever necessary, precise bibliographical information is given in the main text. As the first part of the notes a few books of general interest are listed.

Much of the vocabulary used in mathematics stems from Greek origin, words like "theorem" and "orthogonal." Aside from such familiar terms, there may be a few unfamiliar words. "Scholion" and "anthypharesis" are the two of them that have to be explained most frequently. A *scholion* is a remark to the main text of the *Elements* that was written (mostly in antiquity) in the margin of old manuscripts. Scholia are somewhat like modern footnotes, but they are very valuable because they transmit information from antiquity. *Anthypharesis* (or, equivalently, antaneiresis) is nothing but the Euclidean algorithm for general magnitudes, mostly for line segments.

Finally, the use of the name *Euclid* is ambiguous. On the one hand, it means the author of the *Elements* who lived about 300 B.C.E. On the other hand, "Euclid" stands for a collection of mostly unknown Greek mathematicians who lived between 500 and 300 B.C.E. and contributed most of the material contained in the *Elements* (and had themselves written "Elements" of some kind before Euclid). Used in that sense, a phrase like "Euclid proved . . ." means just "a Greek mathematician proved. . ."