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# Singular Loci of Schubert Varieties

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*With fond memories of Gian-Carlo Rota.*

# Contents

# Preface

This monograph began to take shape in late 1997 when the authors undertook to write a survey article including current results in the theory of singular loci of Schubert varieties. The article quickly became too long for a journal article because of the vast literature on the subject. We felt there was a need to have this diverse collection of results unified in a single source. Hence, we decided to extend the article into a book.

In order to give a broad treatment of the topic we have chosen to include only a limited number of proofs. The proofs included are directly related to the computations of the singular locus. We have included many other results so the reader may comprehend how this subject sits inside the more general topics of algebraic geometry, representation theory and combinatorics.

We have attempted to make this document as accessible as possible to a wide audience. A natural place to begin is with the definitions of the flag manifold, Schubert varieties, Bruhat–Chevalley order and parabolic subgroups. Our main example throughout this text will be the Schubert varieties in the flag manifold $SL_n/B$ (or Type $A$ case). Therefore, we have devoted Chapter 3 to concrete computations on the flag manifold $SL_n/B$, the Grassmannian manifold and their analogs for the other classical groups. Several of the smoothness criteria take particularly nice forms in special cases such as types $A$ and $C$. We have also included extensive tables of the singular locus of a Schubert variety for the Weyl groups of types $A_5$, $B_4$, $C_4$, $D_4$ and $G_2$.

Even though we have included a large amount of background material, this book is not intended to serve as an introduction to root systems, representations of Lie algebras, algebraic groups, Weyl groups or even Schubert varieties. We expect the reader to be somewhat familiar with these subjects, plus have an interest in doing computations with Schubert varieties. We only give a brief review of the definitions and main theorems which will be essential for this text so that the reader should not need to refer to other texts for basic definitions.

This book can be used for a year-long course on Schubert varieties with a main focus on their singularities. The material covered in this book should serve as a good source of information on the singularities of

Schubert varieties for graduate students and researchers working in the area of combinatorics, algebraic geometry, algebraic groups and representation theory.

Throughout the text we have numbered the equations, subsection (or topics), theorems, remarks, etc, in order according to their chapter and section. Therefore, Theorem 2.1.5 is the fifth theorem in the first section of Chapter 2.

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Sara Billey and Venkatramani Lakshmibai
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