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Thomas Hawkins

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Preface

This book is both more and less than a history of the theory of Lie groups during the period 1869–1926. No attempt has been made to provide an exhaustive treatment of all aspects of the theory. Instead, I have focused upon its origins and upon the subsequent development of its structural aspects, particularly the structure and representation of semisimple groups. In dealing with this more limited subject matter, considerable emphasis has been placed upon the motivation behind the mathematics. This has meant paying close attention to the historical context: the mathematical or physical considerations that motivate or inform the work of a particular mathematician as well as the disciplinary ideals of a mathematical school that encourage research in certain directions. As a result, readers will obtain in the ensuing pages glimpses of and, I hope, the flavor of many areas of nineteenth and early twentieth century geometry, algebra, and analysis. They will also encounter many of the mathematicians of the period, including quite a few not directly connected with Lie groups, and will become acquainted with some of the major mathematical schools. In this sense, the book is more than a history of the theory of Lie groups. It provides a different perspective on the history of mathematics between, roughly, 1869 and 1926. Hence the subtitle.

I have divided the book into four parts, each bearing the name of a mathematician. The cast of characters of each part is large, but in each part one mathematician stands out as the central figure. The first part is devoted to the geometrical and analytical origins of the theory of continuous transformation groups of Sophus Lie (1849–1899)—the precursor of the modern theory of Lie groups—and to some of the features of the

theory he went on to develop that are important in what is to follow. In Part II the central figure is Wilhelm Killing (1847–1923), whose interest in the foundations of non-Euclidean geometry led him down a path of research that culminated in his discovery of almost all the central concepts and theorems on the structure and classification of semisimple Lie algebras. Part III is primarily concerned with developments that would now be interpreted as bearing on the representation of Lie algebras, particularly simple and semisimple algebras. Élie Cartan (1869–1951), who began his career by sifting through Killing’s theorems and providing many of the correct ones with rigorous proofs, stands out as the principal figure. In Part IV that role is played by Hermann Weyl (1885–1955), whose contributions to the structure and representation of Lie groups served to bring the earlier developments into something of a completed whole, while at the same time opening up significant new avenues for research. Readers daunted by the size of the entire book should note that each part constitutes a fairly self-contained monograph that can be read independently of the others.

It is my hope that this book will find an audience among students of mathematics in addition to mathematicians, physicists, and historians of mathematics. My experience is that the understanding of a theory is deepened by familiarity both with the considerations that motivated various developments and with the less formal, more intuitive manner in which they were initially conceived. Hence in addition to stressing the motivational, historical context of the mathematics, I have tried to expound the original mathematical conceptions and reasoning clearly and fairly extensively. There are many fine texts devoted to a systematic presentation of the theory as it stands today—too many to list here. Texts from which I have profited include the books by Humphreys, Olver, Samelson, Serre, and Varadarajan included in the list of references.

In composing this first book-length study of the history of Lie groups, I have benefited from the historical writings of many others, as is evident from the citations that follow. Features of this study that I believe are original to me include, in Parts I–II: (1) A detailed analysis of the geometrical and analytical considerations that combined to inspire and inform the creation of Lie’s theory, including as well an assessment of the role of Felix Klein (Chapters 1–2). (2) The first and only careful analysis and assessment of Killing’s papers on Lie algebras and a depiction of the human side of his efforts based upon his extensive correspondence with F. Engel (Chapter 5, Section 6.2). (3) A study of the historical context of Killing’s seminal work in foundations of geometry that takes into consideration the combined effect upon Killing of the contemporaneous developments in the foundations of geometry and his educational background as Weierstrass’s student (Chapter 4).

In Part III I would mention: (4) A portrayal of Cartan’s theory of weights as a gradual but brilliant revamping of Killing’s faulty theory of secondary roots (Sections 6.3–6.4, Sections 8.2 and 8.4,). (5) A study of

the disciplinary ideals of the Paris community of mathematicians at the turn of the century and a discussion within that context of the directions taken by Cartan's research (Chapters 6 and 8). In addition to providing an important part of the historical context of Cartan's work, it makes understandable the twenty-year hiatus between Cartan's thesis work, where we already find the beginnings of his theory of weights and representations, and the memoirs of 1913–14, where the theory is developed fully. (6) An examination of work within Lie's school by mathematicians other than Cartan on what would now be described as dealing with the representation of Lie algebras (Chapter 7). This work is of interest in its own right as an indication of where, to borrow a Kuhnian phrase, the "normal" mathematics within Lie's school was leading. Among other things we find that during 1906–13 G. Kowalewski had devised his own theory of "weights" to deal with a problem equivalent to the one Cartan completely solved in 1913. The comparison of Cartan's work with the quite respectable normal mathematics of Lie's school provides the basis for a genuinely historical appreciation of the magnitude of his genius.

Finally, in Part IV: (7) By his own admission, Weyl was greatly influenced by Hilbert and the climate he fostered at Göttingen. In Chapter 9, drawing upon the extensive researches of other historians, I have attempted to paint a picture of Hilbert's Göttingen that indicates the extent to which Weyl's work on the representation of Lie groups bears the imprint of his experiences there. (8) A careful analysis of Weyl's preoccupation with the mathematics of general relativity, which for the first time makes complete sense out of his remark that "the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups" (Chapter 11).

The book includes a substantial index, which I hope will prove useful. In addition to the usual names of individuals and of mathematical terms, the index lists themes that run through the book, e.g., "conceptual thinking," "generic reasoning," and "local vs. global viewpoint." I have also given names to important theorems so as to facilitate referring to them in the text in a way that will help the reader to recognize them, but if, e.g., the exact statement of what I call Lie's exponential map theorem is desired, the page reference will be found in the index. In addition to theorems, I have also given names to mathematical notions that are not part of the present-day vocabulary (e.g., "Hurwitz integral," "Conditions I–III" of Killing). These terms are also included in the index, and in the case of multiple page references, the one containing the definition is underlined.

In referring to publications listed in the References section, I have followed the standard practice among historians of mathematics. Thus a reference such as [Lie, 1876c:12]—or simply [1876c:12] when authorship is clear—refers to p. 12 of the publication listed under "Lie, S., 1876c" in the reference section. The year may be the year it was submitted or the year of

publication. In the case of someone like Lie, I have used the dates of submission, since his publications were numerous and the submission dates give a better sense of the chronological progression of his mathematical ideas. Both the date of submission and the date of publication are indicated in the reference. For mathematicians whose collected works have been published, the page references generally refer to the original paginations when these are printed in the collected works. When the original pagination is not included in the collected works, I usually give the pagination of the collected works. A glance at the reference, which includes both original and collected works paginations, will make it clear which is being used. The key to the abbreviations used in the text for manuscript archives is given at the beginning of the References section.

More than twenty-five years have passed since I began the research that has made it possible for me to write this book. During that time many individuals and institutions have provided considerable encouragement and assistance. Among institutions I would mention first of all the National Science Foundation, which through its programs in the history of science provided me with vital financial support on several occasions. In particular, I am grateful to Ron Overman, who as program director was always at hand to facilitate my dealings with the NSF. The story that unfolds on the following pages has been enhanced throughout by the use of manuscript sources, and I am indebted to the many individuals at the archives in Göttingen, Oslo, and Zürich (listed at the beginning of the References section) who promptly provided me with copies of requested manuscripts. I am especially indebted to the late Hermann Boerner and to Helga Bertram, of the Mathematisches Institut Giessen, for all they did to make the resources of the Engel archive available to me. Over the years my work on this book has been encouraged in special ways by three departments of mathematics. To my colleagues in the department at Boston University I am grateful for their patient and respectful support of such a long-term project. In particular, I am indebted to Andrew Lyasoff for generously sharing his expertise on $\text{\TeX}$ with me. I also wish to thank the departments at the University of Chicago and Yale University for the special encouragement they provided by periodically inviting me to talk about my latest findings.

Sizable portions of the research behind the book were done while on sabbatical leaves of absence granted by Boston University. I am grateful to the Department of History of Science at Harvard University for arranging for me to be a visiting scholar during 1980–81 and to A. Borel for arranging membership in the School of Mathematics of the Institute for Advanced Study in Princeton during 1988–89. A first draft of the entire book was written during 1996–97 while I was a resident fellow at the Dibner Institute for History of Science and Technology at M.I.T., and I wish to thank the directors and staff of the institute for providing me with a friendly, supportive environment that proved ideal for book-writing. Without that happy year of duty-free tranquillity I am not sure I ever would have managed to bring

all the strands of my research together into a book.

During the many years of researching and writing this book, my attention was focused entirely upon the written word, without any thoughts about a visual component. It was only after the final chapter was written that I realized that it would be especially appropriate in a book such as this to provide the reader with portraits of many of the mathematicians who enter the story. I quickly discovered that locating and obtaining suitable photographs is a difficult, time-consuming task, and I am grateful to the many archives and individuals who assisted me in carrying it out. (The sources of all portraits are given in the References section at the end of the book.) In particular, I want to express my thanks to the following individuals, who far exceeded the expected level professional cooperation and cordiality in their generous efforts to help me: Henri Cartan, Walter Gautschi, Livia Giacardi, Bjorn Jahren, Yvonne Schezt, Arild Stubhaug, and Martha Weyl Thompson.

Finally, I would like to record my gratitude and indebtedness to the following mathematicians and historians who have helped me with their advice and expertise or simply with their encouragement as I labored on this project: Garrett Birkhoff, John Coleman, Leo Corry, Jean Dieudonné, Walter Feit, Jeremy Gray, Bob Hermann, J. E. Humphreys, N. Jacobson, George Mackey, Peter Olver, David Rowe, George Seligman, Jacob Towber, and B. L. van der Waerden. Above all, I am indebted to A. Borel and J.-P. Serre, who throughout the years have encouraged my project and have generously taken the time to read over and comment upon much of what has gone into the book. Except for myself, however, there is no one who has seen and approved everything that is in this final version; the finished product, with whatever faults and idiosyncrasies it may have, is mine alone.

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