Applied Mathematical Sciences

Volume 54

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Wolfgang Wasow

Linear Turning Point Theory

With 19 Illustrations

Springer Science+Business Media, LLC

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AMS Subject Classifications: 65L99, 34E99, 34A30, 34A50

Library of Congress Cataloging in Publication Data

Wasow, Wolfgang Richard,
Linear turning point theory.

(Applied mathematical sciences; vol. 54)

Bibliography: p.

1. Differential equations—Asymptotic theory.

I. Title. II. Series: Applied mathematical sciences
(Springer-Verlag New York Inc.); v. 54.

QA1.A647 vol. 54 510 s 84-20243

[QA372] [513.3'5]

ISBN 978-1-4612-7008-9 ISBN 978-1-4612-1090-0 (eBook)

DOI 10.1007/978-1-4612-1090-0

© 1985 by Springer Science+Business Media New York

Originally published by Springer-Verlag New York Inc. in 1985

Softcover reprint of the hardcover 1st edition 1985

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form without written permission from Springer Science+Business Media, LLC.

Typeset by Computype, Inc., St. Paul, Minnesota.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-7008-9

Preface

My book “Asymptotic Expansions for Ordinary Differential Equations” published in 1965 is out of print. In the almost 20 years since then, the subject has grown so much in breadth and in depth that an account of the present state of knowledge of all the topics discussed there could not be fitted into one volume without resorting to an excessively terse style of writing.

Instead of undertaking such a task, I have concentrated, in this exposition, on the aspects of the asymptotic theory with which I have been particularly concerned during those 20 years, which is the nature and structure of turning points. As in Chapter VIII of my previous book, only linear analytic differential equations are considered, but the inclusion of important new ideas and results, as well as the development of the necessary background material have made this an exposition of book length.

The formal theory of linear analytic differential equations without a parameter near singularities with respect to the independent variable has, in recent years, been greatly deepened by bringing to it methods of modern algebra and topology. It is very probable that many of these ideas could also be applied to the problems concerning singularities with respect to a parameter, and I hope that this will be done in the near future. It is less likely, however, that the analytic, as opposed to the formal, aspects of turning point theory will greatly benefit from such an algebraization. Also, mathematically trained readers who are mostly interested in applications may prefer the more elementary, narrative style in which my previous book was written, and which I have accepted for the present one as well.

The book is primarily concerned with general mathematical methods, but its contents may help applied mathematicians to understand better the many special cases, modifications and additions that can be found in the extensive literature.

The presupposed mathematical background includes the elements of advanced calculus, complex variable theory, linear algebra, and ordinary differential equations, but no knowledge that is not normally acquired by an undergraduate student of mathematics whose main mathematical interest is Analysis.

I am grateful to the Mathematics Department of the University of Wisconsin–Madison, from which I retired four years ago, for allowing me to draw freely on its secretarial services in preparing the manuscript of this book, and special thanks are due to its capable and patient staff of technical typists.

Wolfgang Wasow

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