PART IV
LIE THEORY

The purpose of this final part of the book is threefold.

First of all, we want to complete the program stated in the introduction to Part II. We have completed the first two steps of this program, showing in Part II how the analysis of representations of Lie groups could be reduced to the study of representations of complex Lie algebras, of which the most important are the semisimple; and carrying out in Part III such an analysis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$. To finish the story, we want now to translate our answers back into the terms of the original problem. In particular, we want to deal with representations of Lie groups as well as Lie algebras, and real groups and algebras as well as complex. The passage back to groups is described in Lecture 21, and the analysis of the real case in Lecture 26.

Another goal of this Part is to establish a framework for some of the results of the preceding lectures—to describe the general theory of semisimple Lie algebras and Lie groups. The key point here is the introduction of the Dynkin diagram and its use in classifying all semisimple Lie algebras over $\mathbb{C}$. From one point of view, the impact of the classification theorem is not great: it just tells us that we have in fact already analyzed all but five of the simple Lie algebras in existence. Beyond that, however, it provides a picture and a language for the description of the general Lie algebra. This both yields a description of the five remaining simple Lie algebras and allows us to give uniform descriptions of associated objects: for example, the compact homogeneous spaces associated to simple Lie groups, or the characters of their representations. The classification theory of semisimple Lie algebras is given in Lecture 21; the description in these terms of their representations and characters is given in Lecture 23. The five exceptional simple Lie algebras, whose existence is revealed from the Dynkin diagrams, are studied in Lecture

22; we give a fairly detailed account of one of them ($\mathfrak{g}_2$), with only brief descriptions of the others.

Third, all this general theory makes it possible to answer the main outstanding problem left over from Part III: a description of the multiplicities of the weights in the irreducible representations of the simple Lie algebras. We give in Lectures 24 and 25 a number of formulas for these multiplicities.

This, it should be said, represents in some ways a shift in style. In the previous lectures we would typically analyze special cases first and deduce general patterns from these cases; here, for example, the Weyl character formula is stated and proved in general, then specialized to the various individual cases (this is the approach more often taken in the literature on the subject). In some ways, this is a fourth goal of Part IV: to provide a bridge between the naive exploration of Lie theory undertaken in Parts II and III, and the more general theory readers will find elsewhere when they pursue the subject further.

Finally, we should repeat here the disclaimer made in the Preface. This part of the book, to the extent that it is successful, will introduce the reader to the rich and varied world of Lie theory; but it certainly undertakes no serious exploration of that world. We do not, for example, touch on such basic constructions as the universal enveloping algebra, Verma modules, Tits buildings; and we do not even hint at the fascinating subject of (infinite-dimensional) unitary representations. The reader is encouraged to sample these and other topics, as well as those included here, according to background and interest.