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John Stillwell

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With 165 Figures

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To Wilhelm Magnus

In Memoriam

Preface

Geometry used to be the basis of a mathematical education; today it is not even a standard undergraduate topic. Much as I deplore this situation, I welcome the opportunity to make a fresh start. Classical geometry is no longer an adequate basis for mathematics or physics—both of which are becoming increasingly geometric—and geometry can no longer be divorced from algebra, topology, and analysis. Students need a geometry of greater scope, and the fact that there is no room for geometry in the curriculum until the third or fourth year at least allows us to assume some mathematical background.

What geometry should be taught? I believe that the geometry of surfaces of constant curvature is an ideal choice, for the following reasons:

1. It is basically simple and traditional. We are not forgetting euclidean geometry but extending it enough to be interesting and useful. The extensions offer the simplest possible introduction to fundamentals of modern geometry: curvature, group actions, and covering spaces.

2. The prerequisites are modest and standard. A little linear algebra (mostly 2×2 matrices), calculus as far as hyperbolic functions, basic group theory (subgroups and cosets), and basic topology (open, closed, and compact sets).

3. (Most important.) The theory of surfaces of constant curvature has maximal connectivity with the rest of mathematics. Such surfaces model the variants of euclidean geometry obtained by changing the parallel axiom; they are also projective geometries, Riemann surfaces, and complex algebraic curves. They realize all of the topological types of compact two-dimensional manifolds. Historically, they are the source of the main concepts of complex analysis, differential geometry, topology, and combinatorial group theory. (They are also the source of some hot research topics of the moment, such as fractal geometry and string theory.)

The only problem with such a deep and broad topic is that it cannot be covered completely by a book of this size. Since, however, this is the size of book I wish to write, I have tried to extend my formal coverage in two ways: by exercises and by informal discussions. The exercises include many important and interesting results that could not be fitted into the main text. Where these results are difficult, they have been broken into steps, which I hope are of manageable size. There is an informal discussion at the end of each chapter, sketching historical background, related mathematical ideas, other viewpoints, and generalizations.

Because of their deliberate informality, those discussions may mean different things to different readers, but I hope that they give at least a glimpse of a world vastly larger than the one I have covered formally. Sufficient references are given to enable intrepid readers to explore this world on their own. (References are given in the author [year] format and are collected at the end of the book.) Four references deserve particular mention, as they help to complement my approach. Nikulin and Shafarevich [1987] also extends euclidean geometry by group actions. It goes further in the direction of the third dimension, but not as far into spherical and hyperbolic geometry, or topology because it relies on strictly elementary methods. Jones and Singerman [1987] gives a geometric view of complex analysis, obtaining analytically some of the results we obtain by algebraic and geometric methods. Magnus [1974] and Gray [1986] salvage the historical treasures of the subject, Magnus from group theory and Gray from the theory of differential and algebraic equations.

This book owes a lot to my late teacher, Carl Moppert, who introduced me to the elegant reflection arguments that are common to euclidean, spherical, and hyperbolic geometry. Moppert was a student of Ostrowski, who was a student of Klein, so it may be hoped that some of Klein's spirit lives in the present book. My own students Paul Candler, Matthew Drummond, Greg Findlow, Chris Hough, Thomas Lumley, Monica Mangold, Tony Mason, Jane Paterson, Dawn Tse, and Axel Wabenhorst have also made important contributions in their turn, finding many mistakes and suggesting improvements. Extra thanks are due to Abe Shenitzer, who made a thorough critique of the manuscript just before final revision, and to Anne-Marie Vandenberg and Gertrude Nayak for typing.

I also hope to transmit some of the spirit of Magnus, and his teacher Dehn. It was Dehn who first noticed the remarkable combinatorial properties of non-euclidean tessellations and saw how they simultaneously solve problems in group theory and topology. Wilhelm Magnus not only made great contributions to group theory himself, but also brought the geometric methods to a wider audience. His death just as this book was being completed was sad for all who knew him, but happily his ideas live on and are growing in vigor.

Clayton, Victoria, Australia John Stillwell

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