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(continued after index)

Joel H. Shapiro

Composition Operators

and Classical Function Theory

With 17 Illustrations

Springer Science+Business Media, LLC

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for Jane

Preface

The study of composition operators links some of the most basic questions you can ask about linear operators with beautiful classical results from analytic-function theory. The process invests old theorems with new meanings, and bestows upon functional analysis an intriguing class of concrete linear operators. Best of all, the subject can be appreciated by anyone with an interest in function theory or functional analysis, and a background roughly equivalent to the following twelve chapters of Rudin's textbook *Real and Complex Analysis* [Rdn '87]: Chapters 1–7 (measure and integration, L^p spaces, basic Hilbert and Banach space theory), and 10–14 (basic function theory through the Riemann Mapping Theorem).

In this book I introduce the reader to both the theory of composition operators, and the classical results that form its infrastructure. I develop the subject in a way that emphasizes its geometric content, staying as much as possible within the prerequisites set out in the twelve fundamental chapters of Rudin's book.

Although much of the material on operators is quite recent, this book is not intended to be an exhaustive survey. It is, quite simply, an invitation to join in the fun. The story goes something like this.

The setting is the simplest one consistent with serious “function-theoretic operator theory:” the unit disc U of the complex plane, and the Hilbert space H^2 of functions holomorphic on U with square summable power series coefficients. To each holomorphic function φ that takes U into itself we associate the *composition operator* C_φ defined by

$$C_\varphi f = f \circ \varphi \quad (f \text{ holomorphic on the disc}),$$

and set for ourselves the goal of discovering the connection between the function theoretic properties of φ and the behavior of C_φ on H^2 .

Indeed, it is already a significant accomplishment to show that each composition operator takes H^2 into itself; this is essentially Littlewood's famous Subordination Principle. Further investigation into properties like compactness, spectra, and cyclicity for composition operators leads naturally to classical results like the Julia-Carathéodory Theorem on the angular derivative, the Denjoy-Wolff iteration theorem, Königs's solution of Schröder's functional equation, the Koebe Distortion Theorem, and to hidden gems like the Linear Fractional Model Theorem, and Littlewood's "Counting Function" generalization of the Schwarz Lemma. I list below a more detailed outline.

Chapter 0. This is a prologue, to be consulted as needed, on the basic properties and classification of linear fractional transformations. Linear fractional maps play a vital role in our work, both as agents for changing coordinates and transforming settings (e.g., disc to half-plane), and as a source of examples that are easily managed, yet still rich enough to exhibit surprisingly diverse behavior. This diversity of behavior foreshadows the Linear Fractional Model Theorem, which asserts that every univalent self-map of the disc is conjugate to a linear fractional self-map of some, usually more complicated, plane domain.

Chapter 1. After developing some of the basic properties of H^2 , we show that every composition operator acts boundedly on this Hilbert space. As pointed out above, this is essentially Littlewood's Subordination Principle. We present Littlewood's original proof—a beautiful argument that is perfectly transparent in its elegance, but utterly baffling in its lack of geometric insight. Much of the sequel can be regarded as an effort to understand the geometric underpinnings of this theorem.

Chapters 2 through 4: Having established the boundedness of composition operators, we seek to characterize those that are *compact*. Chapter 2 sets out the motivation for this problem, and in Chapter 3 we discover that the geometric soul of Littlewood's Theorem is bound up in the Schwarz Lemma. Armed with this insight, we are able to characterize the *univally induced* compact composition operators, obtaining a compactness criterion that leads directly to the *Julia-Carathéodory Theorem* on the angular derivative. In Chapter 4 we prove the Julia-Carathéodory Theorem in a way that emphasizes its geometric content, especially its connection with the Schwarz Lemma. At the end of Chapter 4 we give further applications to the compactness problem.

Chapters 5 and 6: Closely related to the Julia-Carathéodory Theorem is the Denjoy-Wolff Iteration Theorem (Chapter 5), which plays a fundamental role in much of the further theory of composition operators. In Chapter 6 we use the Denjoy-Wolff Theorem as a tool in determining the

spectrum of a compact composition operator. Our work on the spectrum leads to a connection between the Riesz Theory of compact operators and Königs's classical work on holomorphic solutions of Schröder's functional equation—the eigenvalue equation for composition operators. We develop the relevant part of the Riesz Theory, and show how, in helping to characterize the spectrum of a compact composition operator, it also provides a growth restriction on the solutions of Schröder's equation for the inducing map. When the inducing map is univalent this growth restriction has a compelling geometric interpretation, which re-emerges in Chapter 9 to provide, under appropriate hypotheses, a geometric characterization of compactness based on linear fractional models.

Chapter 7 and 8: The study of eigenfunctions begun in Chapter 6 leads in two directions. In these chapters we show how it suggests the idea of using simple linear fractional transformations as “models” for holomorphic self-maps of the disc. These *linear fractional models* provide an important tool for investigating other properties of composition operators. We illustrate their use in the study of *cyclicity*. Taking as inspiration earlier work of Birkhoff, Seidel, and Walsh, we show that certain composition operators exhibit *hypercyclicity*; they have a dense orbit (in the language of dynamics, they are *topologically transitive*). Our method involves characterizing the hypercyclic composition operators induced by linear fractional self-maps of U (Chapter 7), and then transferring these results, by means of an appropriate linear fractional model, to more general classes of inducing maps (Chapter 8).

Chapter 9: The second direction for the study of eigenfunctions involves turning around the growth condition discovered in Chapter 6 to provide a compactness criterion: If φ obeys reasonable hypotheses, and the solutions of Schröder's equation for φ do not grow too fast, then C_φ is compact. The requisite growth condition has a simple geometric interpretation in terms of the linear fractional model for φ . Again the key to this work is the Schwarz Lemma, this time in the form of hyperbolic distance estimates that follow from the Koebe Distortion Theorem.

Chapter 10: The solution to the compactness problem for general composition operators leads into value distribution theory, most notably the asymptotic behavior of the Nevanlinna Counting Function. Here our story comes full circle: the crucial inequality on the Counting Function is due to Littlewood, and when reinterpreted it becomes a striking generalization of the Schwarz Lemma.

As you might guess, the expository rules announced above occasionally get bent, but most of these transgressions can be remedied by referring to other parts of [Rdn '87]. The radial limit theorem for H^2 functions is introduced in Chapter 2, although my defense here is that its real application involves only the bounded case, which you find in the legally admissible

part of [Rdn '87]. For Chapter 3 the reader is hoped to have acquired some of the lore concerning compactness and equicontinuity in metric spaces, and to have written down, at some point, the metric for uniform convergence on compact subsets of the disc. Chapters 6, 8 and 10 utilize a standard convergence theorem for infinite products of holomorphic functions that occurs in the first few pages of [Rdn '87, Ch. 15].

The Carathéodory Theorem concerning boundary continuity of Riemann maps onto Jordan domains makes a couple of appearances; but note that it is “almost” proved in Chapter 14 of [Rdn '87]. The discussion of cyclicity for the “non-linear-fractional” case requires Walsh’s theorem on polynomial approximation on Jordan domains, which is a special case of Mergelyan’s Theorem (Chapter 20 of [Rdn '87]). Although the Linear Fractional Model Theorem is discussed in some generality, only those parts needed for the results presented here are proved.

Acknowledgments

This book originated from lectures I gave in March 1991 at Universidad Simon Bolivar in Caracas, and from a longer series of talks at the University of Hawaii spanning the Fall semester of that year. I am particularly grateful to Professor Julio Cano, who arranged my visit to Caracas, and to my colleagues at Hawaii who provided a lively and appreciative audience. In Honolulu I had the good fortune to interact with Wayne Smith and David Stegenga; and from our discussions emerged a joint project, part of which forms Chapter 9 of this book. During this period I also benefited from conversations with Tom Hoover, H.M. Hilden, Marvin Ortel, and H.S. Bear.

Over the past seven years I have had the privilege of collaborating on problems involving cyclicity and composition operators with Paul Bourdon of Washington and Lee University. The material of Chapters 7 and 8 is joint work with him, and my continuing interest in the subject owes a lot to his insight and enthusiasm.

My conversations with Bourdon, Smith, and Stegenga convinced me that the study of composition operators could serve as a showcase for the classical theorems that make the subject so appealing. Steven Krantz started me thinking about writing a book long before I felt ready to undertake it, and he ultimately provided the vital encouragement which got the project off the ground. Paul Bourdon and Wayne Smith read the original manuscript, caught many errors, and suggested numerous changes that greatly improved the exposition. I am deeply indebted to all these people for their support.

East Lansing, Michigan
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Notation and Terminology

Here is a summary of the major notational and linguistic conventions used throughout the book.

Disc and Half-plane. The symbol U denotes the unit disc of the complex plane, often simply called “the disc,” $\bar{U}$ denotes the closed unit disc, and ∂U the unit circle. The symbol φ always denotes a holomorphic self-mapping of U . Similarly, Π denotes the open right half-plane $\{w : \operatorname{Re} w > 0\}$, and $\bar{\Pi}$ is its closure in the plane.

The space of holomorphic functions. We denote by $H(U)$ the space of all holomorphic functions on U , and this space is always understood to be endowed with the topology κ of uniform convergence on compact subsets of U . The notation $f_n \xrightarrow{\kappa} f$ means that the sequence $\{f_n\}$ of functions converges to f uniformly on every compact subset of U .

For functions f holomorphic on U we adopt the notation $\hat{f}(n)$ for the n -th coefficient in the power series expansion of f about the origin:

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad (z \in U).$$

Hilbert space. “Hilbert space” always means “separable Hilbert space.” The norm in any Hilbert space is denoted by “ $\| \cdot \|$,” and the inner product by “ $\langle \cdot, \cdot \rangle$.” The reader is assumed to be familiar with these Hilbert spaces: The usual Lebesgue space L^2 , of (equivalence classes of) measurable functions on the the unit circle, square integrable with respect to arc-length measure; and ℓ^2 , the space of square summable (one-sided) complex sequences.

Operators. The term *operator* always means *bounded linear transformation*, and *finite-rank operators* are bounded linear transformations with finite-dimensional range. When no confusion results, the symbol $\|\cdot\|$, originally used for the Hilbert space norm, will also be used for the norm of a bounded linear operator, or for any other norm that we want to discuss.

Special automorphisms. We will frequently employ the linear fractional transformation

$$\alpha_p(z) \stackrel{\text{def}}{=} \frac{p-z}{1-\bar{p}z}$$

for $p \in U$. This is the *special automorphism* of U that interchanges the origin and the point p (see §4 of Chapter 0 below for more details).

Iterates. If φ is a holomorphic self-map of U and n is a positive integer, then the n -th *iterate* of φ is the n -fold composition of φ with itself. We always denote this map by φ_n :

$$\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi \quad (n \text{ times}),$$

taking care to warn the reader of any conflicts of notation that arise when iterates and other sequences populate the same argument.

Estimates. We frequently write estimates for non-negative functions that look like

$$A(x) \leq \text{const. } B(x),$$

for some range of x . In such inequalities the constant is always positive and finite, and is allowed to change from one occurrence to another, but it never depends on x . When we write

$$A(x) \approx B(x) \quad (\text{for some range of } x)$$

we mean

$$\text{const. } A(x) \leq B(x) \leq \text{const. } A(x)$$

for the relevant values of x . Frequently occurring instances of this are the simple equivalences:

$$1 - x^2 \approx 1 - x \quad (0 \leq x \leq 1),$$

and

$$1 - x \approx \log \frac{1}{x} \quad (\text{as } x \rightarrow 1-).$$