Undergraduate Texts in Mathematics



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*(continued after index)*

Andrew Browder

# Mathematical Analysis

## *An Introduction*

Springer

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With five illustrations.

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*For Anna*

# Preface

This is a textbook suitable for a year-long course in analysis at the advanced undergraduate or possibly beginning-graduate level. It is intended for students with a strong background in calculus and linear algebra, and a strong motivation to learn mathematics for its own sake. At this stage of their education, such students are generally given a course in abstract algebra, and a course in analysis, which give the fundamentals of these two areas, as mathematicians today conceive them.

Mathematics is now a subject splintered into many specialties and subspecialties, but most of it can be placed roughly into three categories: algebra, geometry, and analysis. In fact, almost all mathematics done today is a mixture of algebra, geometry and analysis, and some of the most interesting results are obtained by the application of analysis to algebra, say, or geometry to analysis, in a fresh and surprising way. What then do these categories signify? Algebra is the mathematics that arises from the ancient experiences of addition and multiplication of whole numbers; it deals with the finite and discrete. Geometry is the mathematics that grows out of spatial experience; it is concerned with shape and form, and with measuring, where algebra deals with counting. Analysis might be described as the mathematics that deals with the ideas of the infinite and the infinitesimal; more specifically, it is the word used to describe the great web of ideas that has grown in the last three centuries from the discovery of the differential and integral calculus. Its basic arena is the system of real numbers, a mathematical construct which combines algebraic concepts of addition, multiplication, etc., with the geometric concept of a line, or continuum.

There is no general agreement on what an introductory analysis course should include. I have chosen four major topics: the calculus of functions of one variable, treated with modern standards of rigor; an introduction to general topology, focusing on Euclidean space and spaces of functions; the general theory of integration, based on the concept of measure; and the calculus, differential and integral, for functions of several variables, with the inverse and implicit function theorems, and integration over manifolds. Inevitably, much time and effort go into giving definitions and proving technical propositions, building up the basic tools of analysis. I hope the reader will feel this machinery is justified by some of its products displayed here. The theorems of Dirichlet, Liouville, Weyl, Brouwer, and Riemann's Dirichlet principle for harmonic functions, for instance, should need no applications to be appreciated. (In fact, they have a great number of applications.)

An ideal book of mathematics might uphold the standard of economy of expression, but this one does not. The reader will find many repetitions here; where a result might have been proved once and subsequently referred to, I have on occasion simply given the old argument again. My justification is found in communications theory, which has shown mathematically that redundancy is the key to successful communication in a noisy channel. I have also on occasion given more than one proof for a single theorem; this is done not because two proofs are more convincing than one, but because the second proof involves different ideas, which may be useful in some new context.

I have included some brief notes, usually historical, at the end of each chapter. The history is all from secondary sources, and is not to be relied on too much, but it appears that many students find these indications of how things developed to be interesting. A student who wants to learn the material in the early chapters of this book from a historical perspective will find Bressoud's recent book [1] quite interesting.

While this book is meant to be used for a year-long course, I myself have never managed to include everything here in such a course. A year and a half might be reasonable, for students with no previous experience with rigorous analysis. In different years, I have omitted different topics, always regretfully. Every topic treated here meets one of two tests: it is either something that everybody should know, or else it is just too beautiful to leave out. Nevertheless, life is short, and the academic year even shorter, and anyone who teaches with this book should plan on leaving something out. I expect that most teachers who use this book might also be tempted to include some topic that I have not treated, or develop further some theme that is touched on lightly here.

Those students whose previous mathematical experience is mostly with calculus are in for some culture shock. They will notice that this book is only about 30% as large as their calculus text, and contains about one-tenth as many exercises. (So far, so good.) But it will quickly become clear that some of the problems are quite demanding; I believe that (certainly at this level) more is learned by spending hours, if necessary, on a few problems, sometimes a single problem, than in routinely dispatching a dozen exercises, all following the same pattern. I hope the reader is not discouraged by difficulty, but rewarded by difficulty overcome.

This book consists of theorems, propositions, and lemmas (these words all mean the same thing), along with definitions and examples. Most of these are set off formally as Theorem, Proposition, etc., but some definitions, examples, and theorems are in fact sneaked into the text between the formally announced items. I have been persuaded to number the Theorems, Examples, Definitions, etc. by one sequence. Thus Example 4.4 refers to the fourth item in Chapter 4, where an item could be either a Theorem, Proposition, Lemma, Definition, or Example. It would have been more logical to have it refer to the fourth example in this chapter, but it would have made navigation more difficult.

One of the important things that one learns in a course at this level is how to write a mathematical proof. It is quite difficult to prescribe what constitutes a proper proof. It should be a clear and compelling argument, that forces a reader (who has accepted previous theorems and understands the hypotheses) to accept its assertions. It should be concise, but not cryptic; it should be detailed, but not verbose. We learn to do it by imitating models. Here are two models from ancient Greece. Throughout this book, the symbol ▮ will mark the conclusion of a proof.

**Theorem.** *There are infinitely many primes.*

**Proof.** If $p_1, p_2, \ldots, p_n$ are primes, let $N = p_1 p_2 \cdots p_n + 1$. Then $N$ is not divisible by any $p_j$, $j = 1, 2, \ldots, n$, so either $N$ is a prime, or $N$ is divisible by some prime other than $p_1, p_2, \ldots, p_n$. In either case, there are at least $n + 1$ primes. ▮

Note that this proof assumes a knowledge of what a prime number is, and a previously obtained result that every integer $N > 1$ is divisible by some prime. Note also that the last sentence of the proof has been omitted. It might be either to the effect that the hypothesis that the set of all primes can be listed in a finite sequence $p_1, p_2, \ldots, p_n$ has led to a contradiction, or that we have given the recipe for finding a new prime for each natural number, so that the sequence 2, 3, 7, 43, ..., can be extended indefinitely.

**Theorem.** *The square on the hypotenuse of a right triangle is the sum of the squares on the two shorter sides.*

**Proof.** If the sides of the triangle are $a$, $b$, and $c$, with $c$ the hypotenuse, then the square of side $a + b$ can be dissected in two ways, as shown below. Removing the four copies of the triangle present in each dissection, the theorem follows. ▮

This is one of the rare occasions when I would accept a picture as a proof. (Having once been shown the proof, by a carefully drawn diagram,

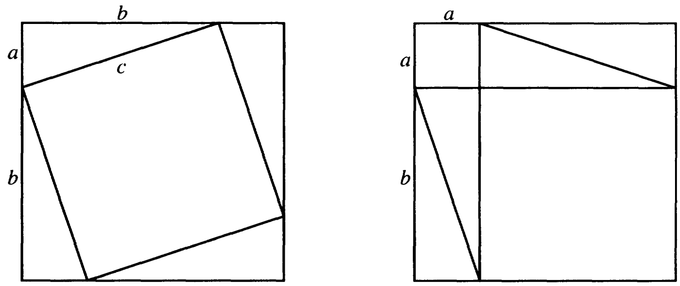


Figure 1. The Pythagorean Theorem.

that every triangle is isosceles, I could never again believe that pictures don't lie.) In this argument, the picture is clear and convincing. By the way, there are few pictures in what follows, and they are all simple and hand-drawn, meant to serve as a guide to simple ideas that are (unfortunately but necessarily) being expressed in awkward or complicated notation. I would urge the readers (of this or any other mathematics) to make their own sketches at all times, preferably crude and schematic.

**Acknowledgments.** It is impossible to list all the writers and individuals who have influenced me in the writing of this book, but for me the model of analysis textbooks at this level has always been Rudin's *Principles of Mathematical Analysis* [11]. The second half of this book (which was written first) was greatly influenced by Spivak's *Calculus on Manifolds* [13], the first clear and simple introduction to Stokes' theorem in its modern form. I was fortunate to read in manuscript Munkres' excellent book *Analysis on Manifolds* [10] while I was writing that earlier version, and profited from it.

Many students found typographical errors and other infelicities in the class notes which formed the first version of the first half of the book, and I want to thank especially Andrew Brecher, Greg Friedman, Ezra Miller, and Max Minzner for their detailed examination, and their suggestions for that part. Eva Kallin made many useful criticisms of an earlier version of the notes on which the second half of this book is based. I am grateful to Xiang-Qian Chang, who read the entire manuscript, and pointed out a great number of rough places.

The mistakes that remain are all my own. I would appreciate hearing about them.

*22 August 2000*

With this second printing, I have had the opportunity to correct a number of typographical errors, infelicities, rough spots, and mistakes. I thank the following for pointing them out to me: William Beckner, Mark Bruso, Xiang-Qian Chang, Eva Kallin, Tom Koornwinder, and Mark McKee.

A.B.

# Contents