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(continued after index)

M.I. Freidlin A.D. Wentzell

Random Perturbations of Dynamical Systems

Second Edition

Translated by Joseph Szücs

With 33 Illustrations



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Preface to the Second Edition

The first edition of this book was published in 1979 in Russian. Most of the material presented was related to large-deviation theory for stochastic processes. This theory was developed more or less at the same time by different authors in different countries. This book was the first monograph in which large-deviation theory for stochastic processes was presented. Since then a number of books specially dedicated to large-deviation theory have been published, including S. R. S. Varadhan [4], A. D. Wentzell [9], J.-D. Deuschel and D. W. Stroock [1], A. Dembo and O. Zeitouni [1]. Just a few changes were made for this edition in the part where large deviations are treated. The most essential is the addition of two new sections in the last chapter. Large deviations for infinite-dimensional systems are briefly considered in one new section, and the applications of large-deviation theory to wave front propagation for reaction-diffusion equations are considered in another one.

Large-deviation theory is not the only class of limit theorems arising in the context of random perturbations of dynamical systems. We therefore included in the second edition a number of new results related to the averaging principle. Random perturbations of classical dynamical systems under certain conditions lead to diffusion processes on graphs. Such problems are considered in the new Chapter 8. Some new results concerning fast oscillating perturbations of dynamical systems with conservation laws are included in Chapter 7. A few small additions and corrections were made in the other chapters as well. We would like to thank Ruth Pfeiffer and Fred Torcaso for their help in the preparation of the second edition of this book.

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Preface

Asymptotical problems have always played an important role in probability theory. In classical probability theory dealing mainly with sequences of independent variables, theorems of the type of laws of large numbers, theorems of the type of the central limit theorem, and theorems on large deviations constitute a major part of all investigations. In recent years, when random processes have become the main subject of study, asymptotic investigations have continued to play a major role. We can say that in the theory of random processes such investigations play an even greater role than in classical probability theory, because it is apparently impossible to obtain simple exact formulas in problems connected with large classes of random processes.

Asymptotical investigations in the theory of random processes include results of the types of both the laws of large numbers and the central limit theorem and, in the past decade, theorems on large deviations. Of course, all these problems have acquired new aspects and new interpretations in the theory of random processes.

One of the important schemes leading to the study of various limit theorems for random processes is dynamical systems subject to the effect of random perturbations. Several theoretical and applied problems lead to this scheme. It is often natural to assume that, in one sense or another, the random perturbations are small compared to the deterministic constituents of the motion. The problem of studying small random perturbations of dynamical systems has been posed in the paper by Pontrjagin, Andronov, and Vitt [1]. The results obtained in this article relate to one-dimensional and partly two-dimensional dynamical systems and perturbations leading to diffusion processes. Other types of random perturbations may also be considered; in particular, those arising in connection with the averaging principle. Here the smallness of the effect of perturbations is ensured by the fact that they oscillate quickly.

The contents of the book consists of various asymptotic problems arising as the parameter characterizing the smallness of random perturbations converges to zero. Of course, the authors could not consider all conceivable schemes of small random perturbations of dynamical systems. In particular, the book does not consider at all dynamical systems generated by random

vector fields. Much attention is given to the study of the effect of perturbations on large time intervals. On such intervals small perturbations essentially influence the behavior of the system in general. In order to take account of this influence, we have to be able to estimate the probabilities of rare events, i.e., we need theorems on the asymptotics of probabilities of large deviations for random processes. The book studies these asymptotics and their applications to problems of the behavior of a random process on large time intervals, such as the problem of the limit behavior of the invariant measure, the problem of exit of a random process from a domain, and the problem of stability under random perturbations. Some of these problems have been formulated for a long time and others are comparatively new.

The problems being studied can be considered as problems of the asymptotic study of integrals in a function space, and the fundamental method used can be considered as an infinite-dimensional generalization of the well-known method of Laplace. These constructions are linked to contemporary research in asymptotic methods. In the cases where, as a result of the effect of perturbations, diffusion processes are obtained, we arrive at problems closely connected with elliptic and parabolic differential equations with a small parameter. Our investigations imply some new results concerning such equations. We are interested in these connections and as a rule include the corresponding formulations in terms of differential equations.

We would like to note that this book is being written when the theory of large deviations for random processes is just being created. There have been a series of achievements but there is still much to be done. Therefore, the book treats some topics that have not yet taken their final form (part of the material is presented in a survey form). At the same time, some new research is not reflected at all in the book. The authors attempted to minimize the deficiencies connected with this.

The book is written for mathematicians but can also be used by specialists of adjacent fields. The fact is that although the proofs use quite intricate mathematical constructions, the results admit a simple formulation as a rule.

Contents

Preface to the Second Edition	v
Preface	vii
Introduction	1
CHAPTER 1	
Random Perturbations	15
§1. Probabilities and Random Variables	15
§2. Random Processes. General Properties	17
§3. Wiener Process. Stochastic Integral	24
§4. Markov Processes and Semigroups	29
§5. Diffusion Processes and Differential Equations	34
CHAPTER 2	
Small Random Perturbations on a Finite Time Interval	44
§1. Zeroth Approximation	44
§2. Expansion in Powers of a Small Parameter	51
§3. Elliptic and Parabolic Differential Equations with a Small Parameter at the Derivatives of Highest Order	59
CHAPTER 3	
Action Functional	70
§1. Laplace's Method in a Function Space	70
§2. Exponential Estimates	74
§3. Action Functional. General Properties	79
§4. Action Functional for Gaussian Random Processes and Fields	92
CHAPTER 4	
Gaussian Perturbations of Dynamical Systems. Neighborhood of an Equilibrium Point	103
§1. Action Functional	103
§2. The Problem of Exit from a Domain	108
§3. Properties of the Quasipotential. Examples	118
§4. Asymptotics of the Mean Exit Time and Invariant Measure for the Neighborhood of an Equilibrium Position	123
§5. Gaussian Perturbations of General Form	132

CHAPTER 5	
Perturbations Leading to Markov Processes	136
§1. Legendre Transformation	136
§2. Locally Infinitely Divisible Processes	143
§3. Special Cases. Generalizations	153
§4. Consequences. Generalization of Results of Chapter 4	157
CHAPTER 6	
Markov Perturbations on Large Time Intervals	161
§1. Auxiliary Results. Equivalence Relation	161
§2. Markov Chains Connected with the Process $(X_t^{\epsilon}, P_x^{\epsilon})$	168
§3. Lemmas on Markov Chains	176
§4. The Problem of the Invariant Measure	185
§5. The Problem of Exit from a Domain	192
§6. Decomposition into Cycles. Sublimit Distributions	198
§7. Eigenvalue Problems	203
CHAPTER 7	
The Averaging Principle. Fluctuations in Dynamical Systems with Averaging	212
§1. The Averaging Principle in the Theory of Ordinary Differential Equations	212
§2. The Averaging Principle when the Fast Motion is a Random Process	216
§3. Normal Deviations from an Averaged System	219
§4. Large Deviations from an Averaged System	233
§5. Large Deviations Continued	241
§6. The Behavior of the System on Large Time Intervals	249
§7. Not Very Large Deviations	253
§8. Examples	257
§9. The Averaging Principle for Stochastic Differential Equations	268
CHAPTER 8	
Random Perturbations of Hamiltonian Systems	283
§1. Introduction	283
§2. Main Results	295
§3. Proof of Theorem 2.2	301
§4. Proofs of Lemmas 3.1 to 3.4	312
§5. Proof of Lemma 3.5	328
§6. Proof of Lemma 3.6	338
§7. Remarks and Generalizations	344
CHAPTER 9	
Stability Under Random Perturbations	361
§1. Formulation of the Problem	361
§2. The Problem of Optimal Stabilization	367
§3. Examples	373

CHAPTER 10

Sharpenings and Generalizations

377

§1. Local Theorems and Sharp Asymptotics

377

§2. Large Deviations for Random Measures

385

§3. Processes with Small Diffusion with Reflection at the Boundary

392

§4. Wave Fronts in Semilinear PDEs and Large Deviations

397

§5. Random Perturbations of Infinite-Dimensional Systems

408

References

417

Index

429