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(continued following index)

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Analysis of Spherical Symmetries in Euclidean Spaces

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Preface

This book gives a new and direct approach into the theories of special functions with emphasis on spherical symmetry in Euclidean spaces of arbitrary dimensions. Essential parts may even be called elementary because of the chosen techniques. The central topic is the presentation of spherical harmonics in a theory of invariants of the orthogonal group.

H. Weyl was one of the first to point out that spherical harmonics must be more than a fortunate guess to simplify numerical computations in mathematical physics. His opinion arose from his occupation with quantum mechanics and was supported by many physicists. These ideas are the leading theme throughout this treatise. When R. Richberg and I started this project we were surprised, how easy and elegant the general theory could be.

One of the highlights of this book is the extension of the classical results of spherical harmonics into the complex. This is particularly important for the complexification of the Funk-Hecke formula, which is successfully used to introduce orthogonally invariant solutions of the reduced wave equation.

The radial parts of these solutions are either Bessel or Hankel functions, which play an important role in the mathematical theory of acoustical and optical waves. These theories often require a detailed analysis of the asymptotic behavior of the solutions. The presented introduction of Bessel and Hankel functions yields directly the leading terms of the asymptotics. Approximations of higher order can be deduced.

These results open the way for many applications in the theories of Fourier integrals and Radon transforms. For example, some of the exponential integrals evaluated for Bessel functions may be used immediately

to formulate the Gauss-Weierstrass integral transforms. The inversion of the Fourier transform is then best verified by a limiting process with these integrals.

It was often regretted that the theory of q -dimensional Fourier transform lacks interesting examples. In this text basic theorems of the Fourier transform naturally provide examples connecting Bessel functions, Hankel functions, spherical harmonics, and their transform.

The presentation of Fourier transforms was included to cover cases needed for applications of the Radon transform. The text closes with integral representations (Nicholson's formula) of the moduli of Hankel functions, which were regarded as particularly important for mathematical optics. In a way, this result is a culmination of all developed topics, because it is achieved by a combination of spherical harmonics, Bessel functions and Hankel functions in the theory of Fourier integrals.

Exercises are introduced partly to stimulate the mathematical ingenuity and partly to bridge the gap between well-known elementary results and their appearance in the new formulations. The book is self-contained and I have tried to limit the amount of machinery of mathematical analysis.

Grateful acknowledgements are due to numerous friends for their kind interest in the work. To Fritz John I owe the suggestion and encouragement to undertake this project. To R. Richberg I am indebted for cooperation, great help, and constructive criticism.

I am grateful to W. Freeden, A. Müller, and G. Bongartz for reading and improving the text of the manuscript. It is also a pleasure to acknowledge the courtesy and the ready cooperation of the publisher.

Claus Müller
Aachen, July 1997

Contents

Preface	v
Introduction	1
§1 Notations and Basic Theorems	1
1 The General Theory	9
§2 Primitive Spaces	10
§3 The Completeness	23
§4 The Funk-Hecke Formula	29
§5 Representations and Interpolations	31
§6 Homogeneous Harmonics	33
2 The Specific Theories	37
§7 The Legendre Polynomials	37
§8 The Laplace Integrals	41
§9 The Gegenbauer Polynomials	44
§10 The Associated Legendre Functions	51
§11 The Associated Spaces $\mathcal{Y}_n^j(q)$	55
§12 Harmonic Differential Operators	61
§13 Maxwell's Theory of Multipoles	67
3 Spherical Harmonics and Differential Equations	75
§14 The Laplace-Beltrami Operators	75
§15 Spherical Harmonics as Eigenfunctions	81

§16	The Legendre Differential Equation	85
§17	The Legendre Functions as Hypergeometric Functions	91
4	Analysis on the Complex Unit Spheres	95
§18	Homogeneous Harmonics in $\mathbb{C}^q$	96
§19	Invariant Integrals on S^{*q-1}	100
§20	Complexification of the Funk-Hecke Formula	108
§21	An Alternative System of Legendre Functions	112
5	The Bessel Functions	119
§22	Regular Bessel Functions	120
§23	Regular Hankel Functions	124
§24	Recursive and Asymptotic Relations	127
§25	Addition Formulas for Hankel Functions of Order Zero . . .	138
§26	Exponential Integrals with Bessel Functions	143
§27	The Traditional Notations	147
6	Integral Transforms	151
§28	Fourier Integrals	152
§29	The Fourier Representation Theorem	154
§30	The Parseval Identity	159
§31	Examples	163
7	The Radon Transform	173
§32	Radon Transforms and Fourier Transforms	173
§33	Radon Transforms and Spherical Symmetries	178
§34	The Nicholson Formulas	186
8	Appendix	195
§35	The Γ -Function	195
§36	The Hypergeometric Function	203
§37	Elementary Asymptotics	207
	References	213
	Index	217