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Integrable Systems

The Verdier Memorial Conference

Actes du Colloque International de Luminy

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Jean-Louis Verdier (1935–1989)

Photograph by Joëlle Pichaud

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Preface

This book constitutes the proceedings of the International Conference on Integrable Systems in memory of J.-L. Verdier. It was held on July 1–5, 1991 at the Centre International de Recherches Mathématiques (C.I.R.M.) at Luminy, near Marseille (France). This collection of articles, covering many aspects of the theory of integrable Hamiltonian systems, both finite- and infinite-dimensional, with an emphasis on the algebro-geometric methods, is published here as a tribute to Verdier who had planned this conference before his death in 1989 and whose active involvement with this topic brought integrable systems to the fore as a subject for active research in France.

The death of Verdier and his wife on August 25, 1989, in a car accident near their country house, was a shock to all of us who were acquainted with them, and was very deeply felt in the mathematics community. We knew of no better way to honor Verdier's memory than to proceed with both the School on Integrable Systems at the C.I.M.P.A. (Centre International de Mathématiques Pures et Appliquées in Nice), and the Conference on the same theme that was to follow it, as he himself had planned them. D. Bennequin, P. Cartier and A. Chenciner agreed to join O. Babelon and Y. Kosmann-Schwarzbach to form a new organizing committee, chaired by P. Cartier. The final list of speakers at the Conference was very close to the original list of invitations discussed with Verdier himself, and the invited participants included ten students chosen from among those who had attended the C.I.M.P.A. School, as originally planned by Verdier.

The refereed articles in this volume represent the advances in the field of complete integrability that were reported at the Luminy conference. In many cases, articles have been updated for publication. In two instances, where the results had been previously published, only summaries with references appear here. The articles represent very diverse methods and report very diverse results. This is a reflection of the complexity and richness of the field of research that is referred to as the theory of completely integrable systems.

In his preliminary text, D. Bennequin takes us into “the garden of integrable systems.” He surveys the evolution of the subject, from Abel onwards, explaining the connections with the classical theory of elliptic functions, describing how algebraic curves and the infinite Grassmannian came to play a prominent role in the theory. He then analyzes the important contributions that Verdier made in this area.

The first part of this book contains articles that make essential use of Riemann surfaces and their theta functions in order to construct classes of solutions of integrable systems, and articles dealing with the tau-functions

that generalize the classical theta functions. The first three papers in this part exemplify the algebro-geometric methods, while the next four deal more specifically with the tau-functions in their various guises.

A. Treibich, who was a close collaborator of Verdier, studies the family of elliptic solitons of the Kadomtsev–Petviashvili hierarchy associated with a projective curve, showing that if Γ is a tangential cover of an elliptic curve E , then its compactified Jacobian covers a symmetric power of E .

In an article written with B. van Geemen, E. Previato, who also collaborated with Verdier, describes recent work on the space of higher-order nonabelian theta functions over a Riemann surface of genus at least two, whose dimension is given by the Verlinde numbers, arising in the fusion rules of conformal field theory, and some connections of nonabelian theta functions with the Schottky problem.

The lecture of H. Knörrer, describing his work with N. Ercolani and E. Trubowitz, deals with the immersed submanifolds of $\mathbb{R}^3$ with constant mean curvature. The link with integrable systems lies in the fact that such immersions can be found by solving the elliptic sinh–Gordon equation, quasi-periodic solutions of which can be constructed by means of the Riemann theta function of hyperelliptic curves.

In the following articles the tau-functions of various integrable systems play a prominent part.

L. Takhtajan’s paper shows that classical modular forms generate a tau-function for several integrable reductions of the self-dual Yang–Mills equations, and therefore general classes of solutions of many equations, in $0 + 1$, $1 + 1$, and $2 + 1$ dimensions.

G. Wilson reviews the generalized Ablowitz–Kaup–Newell–Segur equations associated with a simple Lie algebra, $\mathfrak{g}$, namely the evolution equations equivalent to the zero-curvature equation for a connection obtained from a “bare” connection by the dressing action of the associated loop group. He shows that, when $\mathfrak{g}$ is simply laced, solutions of the $\mathfrak{g}$ AKNS equations can be obtained in terms of tau-functions, which he defines by means of the canonical trivialization of the fibration of the central extension of the loop group over the “big cell” in the loop group.

By redefining the tau-function of Segal and Wilson, L. Dickey is able to determine a tau-function for the hierarchies generated by matrix first-order differential operators which generalize the AKNS equations and for the multi-component KP hierarchies.

P. van Moerbeke studies the blowing-up of a solution of the Korteweg–de Vries equation with respect to a complexified time variable, and, more generally, the compactification of isospectral manifolds of differential operators. He then poses and answers analogous questions for the isospectral families of periodic Jacobi matrices.

In the second part of the book, the main emphasis is on the Hamilton-

ian formalism, and for the last two papers in this part, on the bihamiltonian formalism, while the first one is an illustration of the interplay of the Hamiltonian and the algebro-geometric methods in the study of integrability.

The joint work of N. M. Ercolani, H. Flaschka and S. Singer was presented in Flaschka's lecture on the geometry of the Kostant–Toda lattice, whose Lax matrix has entries below, on and just above the diagonal, the latter being all equal to 1. Using ideas from complex algebraic geometry, they prove the complete integrability of this system, and they determine its constants of motion, which are rational (rather than polynomial) functions on the phase-space.

Studying the vertex set of a Hamiltonian action of a compact commutative Lie group on a compact symplectic manifold, i.e., the image under the moment map of the set of fixed points, V. Guillemin proves a local rigidity theorem for Hamiltonian actions.

In his short contribution, A.T. Fomenko states a conjecture regarding the determination of all the integrable geodesic flows on two-dimensional compact manifolds.

P.J. Olver uses his classification of bihamiltonian structures based on the double Darboux theorem of Turiel in order to draw a list of canonical forms for bihamiltonian systems, and he obtains criteria for both the local and global integrability of such systems.

In the lecture of F. Magri, written in collaboration with P. Casati and M. Pedroni, the theory of soliton equations is presented from the Hamiltonian point of view, the hierarchies of bihamiltonian equations being generated by the Casimir functions of a pencil of Poisson brackets. It is shown that Sato's operator corresponds to the differential of a Casimir function expressed in terms of pseudodifferential operators, and Sato's equations to the vanishing of Poisson brackets on one-forms.

The third part of the book contains two papers that deal with the theory of two-dimensional solvable lattice models in which the quantum Yang–Baxter equation plays a fundamental role.

E. Date's contribution is a summary of his work on the finite-dimensional cyclic representations of the quantum groups at roots of unity and their relation with the two-dimensional lattice models associated with higher genus algebraic curves.

In his lecture, J.-M. Maillard presented his joint work with M. Bellon and C. M. Viallet, showing that the quantum Yang–Baxter equation admits a symmetry group which acts by birational projective transformations on the algebraic varieties which parametrize the solutions of the equation, and he discussed the generalization of these results to the tetrahedron and hyper-simplicial equations that are the analogues of the QYBE for lattices of dimension 3 or more.

In the concluding article, B. Dubrovin shows the interrelations of many aspects of the integrability of the hierarchies of Hamiltonian systems with topological field theory (TFT). He defines the limiting or “averaging” process of a hierarchy and of the tau-function which yields a system of partial equations whose coefficients induce a Frobenius structure on the invariant manifolds of the hierarchy, and hence a solution of the Witten–Dijkgraaf–E. Verlinde–H. Verlinde (WDVV) equations from 2-dimensional TFT. The bihamiltonian structure of the integrable hierarchy is used in the calculation of higher genus corrections, and the WDVV equations are shown to specify the periods of the Abelian differentials of Riemann surfaces as functions on moduli spaces of these surfaces, so that both the bihamiltonian approach and the algebro-geometric methods enter the theory.

The Conference was funded by the C.I.R.M. and by the Société Mathématique de France. Additional support was provided by the French Ministry of Foreign Affairs, and grants were offered by the University of Paris VII, the U.F.R. de Mathématiques et Informatique of the University of Paris VII and the C.N.R.S. research unit U.R.A. 212 (Théories Géométriques). Financial support also came from the Fédération Française des Sociétés d’Assurances and is hereby gratefully acknowledged. We thank in particular our friend and colleague, Prof. M. Flato, who helped us with the fundraising even though he was not free to participate in the Conference.

It is a pleasure to thank the C.I.R.M. and its director, G. Lachaud, for a well-organized and very pleasant conference in the beautiful surroundings of Luminy, and the C.I.M.P.A. and its managing director, J.-M. Lemaire, for providing financial support for the students of the School on Integrable Systems who participated in the Conference.

We thank the editors of the series Progress in Mathematics for offering to publish this volume, and the staff of Birkhäuser Publishing Company for their help and efficient work.

The long-term planning and organizational work were performed with the secretarial assistance of Madame Claudine Roussel, from the University of Paris VII, who joined us at Luminy to handle some of the day-to-day needs of the conference. She also assisted us in the editing of this volume and the forthcoming volume of the courses given at the C.I.M.P.A. school. In the name of all the participants in the conference, we thank her for her expert and invariably cheerful collaboration.

Paris, February 1993

O. Babelon

P. Cartier

Y. Kosmann-Schwarzbach