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The Orbit Method in Geometry and Physics

In Honor of A.A. Kirillov

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A.A. Kirillov

Preface

The volume is dedicated to A.A. Kirillov and emerged from an international conference which was held in Luminy, Marseille, in December 2000, on the occasion of Alexandre Alexandrovitch's 26th birthday. The conference was devoted to the orbit method in representation theory, an important subject that influenced the development of mathematics in the second half of the XXth century. Among the famous names related to this branch of mathematics, the name of A.A. Kirillov certainly holds a distinguished place, as the inventor and founder of the orbit method.

The research articles in this volume are an outgrowth of the Kirillov Fest and they illustrate the most recent achievements in the orbit method and other areas closely related to the scientific interests of A.A. Kirillov.

The orbit method has come to mean a method for obtaining the representations of Lie groups. It was successfully applied by Kirillov to obtain the unitary representation theory of nilpotent Lie groups, and at the end of this famous 1962 paper, it was suggested that the method may be applicable to other Lie groups as well. Over the years, the orbit method has helped to link harmonic analysis (the theory of unitary representations of Lie groups) with differential geometry (the symplectic geometry of homogeneous spaces). This theory reinvigorated many classical domains of mathematics, such as representation theory, integrable systems, complex algebraic geometry. It is now a useful and powerful tool in all of these areas.

The volume contains an overview of A.A. Kirillov's scientific work written by Jacques Dixmier. This general account provides a clear description of the orbit method, its origins and achievements, as well as an outline of other contributions of A.A. Kirillov. The other seventeen research articles published here cover a number of different topics in representation theory: the orbit method in the theory of unitary representations of Lie groups, combinatorial aspects of representation theory, quantization, harmonic analysis, Lie superalgebras and their representations, homological (super)algebras.

A. Alekseev and F. Petrov introduce a "string inspired" functional on the space of representations of the Lie algebra $\mathfrak{su}(2)$ and prove that it achieves its minimum on the set of irreducible unitary representations. A conjecture is proposed that the same is true for any compact Lie algebra.

J. Alev and D.R. Farkas study prime and Poisson ideals in the Poisson algebra of G -invariant polynomials on a complex symplectic vector space for a given finite group G . They provide a complete classification in four dimensions or for an abelian G .

V. Baranovsky, S. Evens and V. Ginzburg analyze the Bernstein–Gelfand–

Gelfand category of representations of the quantum torus algebra. They classify its irreducible objects in terms of semistable principal bundles on elliptic curves.

R. Brylinski provides a noncommutative analog of the Kraft–Procesi model and constructs a Dixmier algebra attached to any nilpotent orbit of a complex Lie group. The associated graded algebra is proved to be a Poisson algebra isomorphic to the coordinate ring on the closure of the orbit.

P. Grozman, D. Leites and I. Shchepochkina study simple Lie superalgebras of polynomial vector fields on superspaces. They list defining relations for four of the known five exceptional Lie superalgebras of this class.

E. Gutkin considers a natural class of induced representations of the Weyl group of some classical groups. He analyzes these representation by means of the classical Schur–Weyl duality in the case of $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$.

A. Joseph and A. Melnikov study a class of Lagrangian subvarieties on nilpotent coadjoint orbits of $SL(n, \mathbb{C})$ called a hypersurface orbital varieties. They give a description of such varieties and construct a simple module with integral highest weight which quantizes this orbit.

D. Kazhdan and A. Polishchuk consider rational representations of a semisimple algebraic Lie group over a p -adic local field. They introduce the notion of ε -stable distributions on this representation and prove that the notions of stable and ε -stable distributions get switched by the Fourier transform.

B. Kostant and P. Michor study and describe the main properties of the (generalized) Cayley map from a connected complex reductive algebraic group to its Lie algebra. They provide an explicit decomposition of the algebra of regular functions on the group into the tensor product of the subalgebra of invariant functions, and a subspace of harmonic functions.

A.A. Kirillov introduces, via some combinatorial properties, the set of coadjoint orbits of the group of strictly upper triangular $(n \times n)$ -matrices with entries in a finite field $\mathbb{F}_q$. The number of coadjoint orbits is computed in terms of a series of polynomial labeled by n , obtained by a variant of the Euler–Bernoulli triangle. The character formula for irreducible representations of the group is obtained by modification of the exponential map. Several conjectures and open problems are formulated.

V. Gorbunov, F. Malikov and V. Schechtman propose a continuation of earlier work on chiral geometry designed to geometrize the theory of vertex operator algebras. They construct chiral differential operators associated with a sheaf of commutative superalgebras over a smooth scheme.

Yu. Neretin describes the algebraic projective compactification of $GL(n, \mathbb{C})$. He also constructs a universal object named the “sea urchin” for all such compactifications.

A. Okounkov discusses the concavity of the logarithm of the multiplicity of an irreducible representation as a function of its highest weight. He proves this property in some special cases and formulates a general conjecture that would be an analog of concavity of entropy in statistical mechanics.

G. Olshanski (with an Appendix by A. Borodin) studies a 2-parametric family of probability measures on the Thoma simplex of factor representations of

the infinite symmetric groups. Probability measures on the Thoma simplex are viewed as random point processes on the real line and their correlation functions are calculated.

W. Rossmann studies smooth toric varieties of a special type, called Bott towers. In the framework of the orbit method, he associates to any such toric variety the character of a unitary representation. Inspired by the Kirillov character formula, he gives two explicit expressions of this character by means of statistical mechanics and path integral calculations.

A. Sergeev gives an interpretation of the projective Schur functions in terms of bispherical functions on certain homogeneous superspaces. These functions are realized as eigenfunctions of differential operators obtained as radial parts of Laplace operators on Lie superalgebras $\mathfrak{gl}(n|n)$ and $\mathfrak{q}(n)$.

I. Shchepochkina classifies maximal non-simple irreducible subalgebras of the classical linear Lie superalgebras, thus generalizing Dynkin's classification in the Lie algebra case.

The articles published in this volume reflect a wide range of A.A. Kirillov's research areas. Nevertheless, some topics could not be presented here, namely infinite-dimensional Lie groups, their orbits and representations, the Virasoro algebra and its applications in mathematical physics and differential geometry.

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December 2002

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