Part II
The Lebesgue Integral

Integration (in geometrical form) goes back to Archimedes [6], but he had practically no followers for almost two millennia. The Newton–Leibniz formula revolutionized the discipline in the seventeenth century, and led to the solution of a great number of geometrical and mechanical problems. A solid theoretical foundation became indispensable, especially after the publication of Fourier's work on heat propagation in [148].

Riemann [371] extended Cauchy's integral [80] to a class of not necessarily continuous functions. Subsequently much research was devoted to the construction of more general integrals and to the simplification of their manipulation. Following the works of Harnack [192, 194], Hankel [190], du Bois-Reymond [52], Jordan [230], Stolz [437] and Cantor [74], Peano [353] introduced the *finitely additive measures*, based on finite covers by intervals or rectangles.

Borel [59] discovered that *countable* covers lead to better, *σ-additive measures.* Baire [16, 17] enlarged the class of continuous functions by the repeated operation of pointwise limits of function sequences. Motivated by the works of Borel and Baire, Lebesgue [287, 288] defined a very general integral. He obtained a much wider class of integrable functions, and at the same time simpler limit theorems than before. He also greatly extended the validity of the Newton–Leibniz formula.

The extraordinary strength of the Lebesgue integral was demonstrated by subsequent important discoveries of Vitali, Beppo Levi, Fatou, Riesz, Fischer, Fréchet, Fubini (1905–1910) and others. These works also led to the development of *Functional Analysis*. The Lebesgue integral later allowed Kolmogorov to give a solid foundation of probability theory [252] and Sobolev to introduce new function spaces for the successful investigation of partial differential equations [426, 427].

F. Riesz gave nice historical accounts in two papers [390, 391]; for more complete surveys we refer to [61, 115, 198, 360–362].

More than a half-century after its publication, the monograph of Riesz and Sz.-Nagy [394] contains still perhaps the most elegant presentation of this theory. We follow this approach, with some minor subsequent improvements. Further results and exercises may be found in the following works: [68, 92, 188, 270, 351, 403, 406, 409, 451].