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Introduction to Tensor Products of Banach Spaces

With 31 Figures

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To my parents

Preface

This book is intended as an introduction to the theory of tensor products of Banach spaces. The prerequisites for reading the book are a first course in Functional Analysis and in Measure Theory, as far as the Radon–Nikodým theorem. The book is entirely self-contained and two appendices give additional material on Banach Spaces and Measure Theory that may be unfamiliar to the beginner. No knowledge of tensor products is assumed.

Our viewpoint is that tensor products are a natural and productive way to understand many of the themes of modern Banach space theory and that "tensorial thinking" yields insights into many otherwise mysterious phenomena. We hope to convince the reader of the validity of this belief.

We begin in Chapter 1 with a treatment of the purely algebraic theory of tensor products of vector spaces. We emphasize the use of the tensor product as a linearizing tool and we explain the use of tensor products in the duality theory of spaces of operators in finite dimensions. The ideas developed here, though simple, are fundamental for the rest of the book.

In Chapter 2 we introduce the projective tensor product. The space of bounded bilinear forms on the product $X \times Y$ of two Banach spaces is the dual of the projective tensor product, $X\hat{\otimes}_\pi Y$. Projective tensor products with ℓ_1 and $L_1(\mu)$ are particularly easy to interpret and the Bochner integral for vector valued functions is introduced. The first hint of the influence of finite dimensional structure appears here with the definition of the $\mathcal{L}_1$-spaces. We give some elementary applications of the Rademacher functions, including the Khinchine Inequality and its implications for the structure of tensor products of $L_p(\mu)$ spaces. Finally, we introduce the class of nuclear operators.

The injective tensor product is studied in Chapter 3, along with the class of integral operators. The attempt to represent the injective tensor product $L_1(\mu)\hat{\otimes}_\varepsilon X$ as a space of vector valued functions leads to the definition of the Pettis integral and among the applications we prove the Orlicz–Pettis theorem, which provides a bridge between weak and norm summability. The importance of the class of integral operators derives in part from the fact that certain canonical operators, such as the injection of $L_\infty(\mu)$ into $L_1(\mu)$, where μ is a finite measure, are of this type. We study this idea in detail.

Chapter 4 is devoted to the approximation property and related matters. The chapter begins with a list of open problems from the previous chapters, all of which depend on the presence of the approximation property in some form for their solution. We give a detailed account of the property and its applications. We then turn to the question of when the projective, or injective, tensor product of reflexive spaces is also reflexive. The chapter concludes with an examination of the construction of tensor product bases for spaces that have Schauder bases.

The Radon–Nikodým property, and its role in the theory of tensor products, is the subject of Chapter 5. We give a complete account of the vector measure theory needed for this purpose. Projective and injective tensor products with the space of measures are characterized in terms of spaces of vector measures. We develop the representation theory of operators on $C(K)$ spaces by means of vector measures, culminating in the phenomenon of the coincidence of the classes of integral and nuclear operators in the presence of the Radon–Nikodým property. The possession of this property by reflexive Banach spaces and separable duals, among others, is established through a reformulation of the property in terms of the representability of operators on $L_1(\mu)$ spaces. We present several striking applications that illustrate the power of the Radon–Nikodým property, finishing with the Principle of Local Reflexivity.

Chapter 6 begins with a discussion of uniform crossnorms and the central concept of a tensor norm. The Chevet–Saphar norms g_p and d_p are introduced, leading to a study of the class of p-summing operators. Among the results treated here are the Pietsch Domination Theorem for p-summing operators. The highlight of the chapter is the Grothendieck inequality. After proving this fundamental result, we give some of its applications to operators between classical Banach spaces.

In Chapter 7 we embark on a thorough investigation of the basic properties of tensor norms. We begin with Schatten's construction of a dual norm, pointing out its shortcomings and the benefits to be obtained from adopting the Grothendieck definition. The question of when the two duals coincide leads in a natural way to the concept of accessibility for a tensor norm. After examining the interconnections between accessibility, duality and the possession of injective or projective properties, we study the injective and projective associate norms that can be generated from a tensor norm. This machinery enables us to identify the duals of the Chevet–Saphar tensor norms and this in turn leads to the classes of p-integral operators. We then turn our attention to the Hilbertian tensor norm, w_2, and the central role it plays in the theory by virtue of the Grothendieck Inequality. After an account of the associated classes of 2-factorable, or Hilbertian, and 2-dominated operators, we conclude with Grothendieck's classification of the fourteen "natural" tensor norms.

In Chapter 8 we present a very brief introduction to the theory of Operator Ideals, concentrating on the connection with the parallel theory of Tensor Norms. Each tensor norm α generates two Banach ideals, the ideal of α-nuclear operators and the ideal of α-integral operators, which coincide in finite dimensions. We show that these ideals are the smallest and largest ideals respectively that are associated with α. We also look at the relationship between a Banach ideal and the tensor norm associated with it.

Each chapter is accompanied by a set of exercises. They are, for the most part, straightforward applications of the subject matter of the chapter in question, and are designed to help the reader in coming to terms with the concepts introduced there. However, we have resisted the temptation to use the exercises as a device for introducing topics that could not be accommodated in the text for reasons of space.

We make no claims whatever to originality, except perhaps in the arrangement of the material. Our overriding objective has been to provide a self-contained, accessible and compact introduction to the subject. We have been highly selective in our choice of topics, preferring to spend time on a small set of important themes rather than include everything one might like to have seen. The expert will find that many of his or her favourite areas have been omitted. We hope that the reader will go on to study the more comprehensive works mentioned in Appendix A. Most of all, we hope that the reader will enjoy this book.

It is a great pleasure to thank all those who have helped me in one way or another. My colleagues in the Mathematics Department in Galway have always been supportive. Departmental heads Ted Hurley and Martin Newell provided crucial practical assistance. My students, Pádraig Kirwan and Bogdan Grecu, learned much of this material with me and read many early drafts. I am grateful to Raymundo Alencar, Richard Aron, Juan Bes, Fernando Blasco, Joe Diestel, Klaus Floret, Mikael Lindström, Manuel Maestre, Yannis Sarantopoulos, Andrew Tonge, Barry Turret and Ignacio Zalduendo for many helpful conversations and for sharing their knowledge with me. Seán Dineen provided vital encouragement and advice at critical moments. Michael Mackey, Niall Madden, Götz Pfeiffer and Dirk Werner all gave invaluable assistance with TEX-related matters. Jose Ansemil and Chris Boyd gave generously of their time in reading drafts of the book and providing many corrections, suggestions and ideas. Dirk Werner not only read and corrected many drafts, but was also a knowledgeable and attentive sounding board. Without his deep knowledge and experience, this would have been a much smaller book.

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Galway, September 2001 *Ray Ryan*

Notation and Terminology

The letters X, Y, Z will denote Banach spaces over the scalar field $\mathbb{K}$, which may be either the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$. The letters E, F, M, N will usually denote finite dimensional Banach spaces. The closed unit ball of a Banach space X will be denoted by B_X.

The term *operator* will mean a bounded linear mapping. The space of operators from X into Y will be denoted by $\mathcal{L}(X,Y)$ and the dual space of X by X^*. Elements of a dual space X^* will typically be denoted by φ or ψ and elements of a bidual X^{**} by symbols such as x^{**} or y^{**}. If φ is a linear functional on X, we shall denote the value of φ at an element x of X either by $\varphi(x)$ or $\langle x, \varphi\rangle$.

We shall denote the space of bounded bilinear forms on the product $X \times Y$ of two Banach spaces by $\mathcal{B}(X \times Y)$, with the norm given by $\|B\| = \sup\{|B(x,y)| : x \in B_X, y \in B_Y\}$.

$C(K)$ and $L_p(\mu)$ will denote the usual sequence spaces, where $1 \leq p \leq \infty$. We shall follow the traditional abuse of notation in using a symbol such as f to denote both the equivalence class of a function in $L_p(\mu)$ and the function itself.

The symbols c_0, ℓ_p, will denote the usual sequence spaces and ℓ_p^n will denote the space $\mathbb{K}^n$ with the ℓ_p-norm. If X is a Banach space, the space $\ell_p(X)$ consists of all sequences $x = (x_n)$ in X for which the scalar sequence $(\|x_n\|)$ belongs to ℓ_p, with the norm of x defined to be the ℓ_p-norm of $(\|x_n\|)$. A similar remark applies to the space $c_0(X)$. The scalar sequence spaces obtained when the indexing set $\mathbb{N}$ is replaced by a set I will be denoted by $c_0(I)$ and $\ell_p(I)$; the context will remove any possible notational confusion with the above-mentioned vector valued sequence spaces.

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