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(continued after index)

Steven Roman

Introduction to the Mathematics of Finance

*From Risk Management
to Options Pricing*

With 55 Figures

Springer

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To Donna

Preface

This book covers two main areas of mathematical finance. One is portfolio risk management, culminating in the Capital Asset Pricing Model and the other is asset pricing theory, culminating in the Black–Scholes option pricing formula. Our discussion of portfolio risk management takes but a single chapter. The rest of the book is devoted to the study of asset pricing models, which is currently a subject of great interest and much research.

The intended audience of the book is upper division undergraduate or beginning graduate students in mathematics, finance or economics. Accordingly, no measure theory is used in this book.

I realize that the book may be read by people with rather diverse backgrounds. On the one hand, students of mathematics may be well prepared in the ways of mathematical thinking but not so well prepared when it comes to matters related to finance (portfolios, stock options, forward contracts and so on). On the other hand, students of finance and economics may be well versed in financial topics but not as mathematically minded as students of mathematics.

Since the subject of this book is the *mathematics* of finance, I have not watered down the mathematics in any way (appropriate to the level of the book, of course). That is, I have endeavored to be mathematically rigorous *at the appropriate level*. On the other hand, the reader is not assumed to have any background in finance, so I have included the necessary background in this area (stock options and forward contracts).

I have also made an effort to make the book as mathematically self-contained as possible. Aside from a certain comfort level with mathematical thinking, a freshman/sophomore course in linear algebra is more than enough. In particular, the reader should be comfortable with matrix algebra, the notion of a vector space and the kernel and range of a linear transformation. The method of Lagrange multipliers is used in a couple of proofs related to risk management, but these proofs can be skimmed or omitted if desired.

Of course, probability theory is ever present in the area of mathematical finance. In this respect, the book is self-contained. Several chapters on probability theory are placed at appropriate places throughout the book.

The idea is to provide the necessary theory on a “need-to-know” basis. In this way, readers who choose not to cover the continuous pricing theory, for example, need not deal with matters related to continuous probability.

The book is organized as follows. The first chapter is devoted to the elements of discrete probability. The discussion includes such topics as random variables, independence, expectation, covariance and best linear predictors. If readers have had a course in elementary probability theory then this chapter will serve mostly as a review.

Chapter 2 is devoted to the subject of portfolio theory and risk management. The main goal is to describe the famous Capital Asset Pricing Model (CAPM). The chapter stands independent of the remainder of the book and can be omitted if desired.

The remainder of the book is devoted to asset pricing models. Chapter 3 gives the necessary background on stock options. In Chapter 4, we briefly illustrate the technique of asset pricing through the assumption of no arbitrage by pricing plain-vanilla forward contracts and discussing some simple issues related to option pricing, such as the *put-call option parity formula*, which relates the price of a put and a call on the same underlying asset with the same strike price and expiration time.

Chapter 5 continues the discussion of discrete probability, covering conditional probability along with more advanced topics such as partitions of the sample space and knowledge of random variables, conditional expectation (with respect to a partition of the sample space) stochastic processes and martingales. This material is covered at the discrete level and always with a mind to the fact that it is probably being seen by the student for the first time.

With the background on probability from Chapter 5, the reader is ready to tackle discrete-time models in Chapter 6. Chapter 7 describes the Cox–Ross–Rubinstein model. The chapter is short, but introduces the important topics of drift, volatility and random walks.

Chapter 8 introduces the very basics of continuous probability. We need the notions of convergence in distribution and the Central Limit Theorem so that we can take the limit of the Cox–Ross–Rubinstein model as the length of the time periods goes to 0. We perform this limiting process in Chapter 9 to get the famous Black–Scholes Option Pricing Formula.

In Chapter 10 we discuss optimal stopping times and American options. This chapter is perhaps a bit more mathematically challenging than the previous chapters.

There are two appendices in the book, both of which are optional. In Appendix A, we discuss the problem of pricing nonattainable alternatives in a discrete model. The material may be read anytime after reading Chapter 6. Appendix B covers background information on convexity that is used in Chapter 6.

A Word on Definitions

Unlike many areas of mathematics, the subject of this book, namely, the mathematics of finance, does not have an extensive literature at the undergraduate level. Put more simply, there are very few undergraduate textbooks on the mathematics of finance.

Accordingly, there has not been a lot of precedent in setting down the basic theory at the undergraduate level, where pedagogy and use of intuition are (or should be) at a premium. One area in which this seems to manifest itself is the lack of terminology to cover certain situations.

Accordingly, on rare occasions I have felt it necessary to invent new terminology to cover a specific concept. Let me assure the reader that I have not done this lightly. It is not my desire to invent terminology for any other reason than as an aid to pedagogy.

In any case, the reader will encounter a few definitions that I have labeled as *nonstandard*. This label is intended to convey the fact that the definition is not likely to be found in other books nor can it be used without qualification in discussions of the subject matter outside the purview of this book.

Thanks Be to...

Finally, I would like to thank my students Lemee Nakamura, Tristan Igualada and Christopher Lin for their patience during my preliminary lectures and for their helpful comments about the manuscript. Any errors in the book, which are hopefully minimal, are my responsibility, of course. The reader is welcome to visit my web site at www.romanpress.com to learn more about my books or to leave a comment or suggestion.

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Notation Key and Greek Alphabet

$\langle, \rangle$ inner product (dot product) on $\mathbb{R}^n$
 $\mathbf{1}$ the unit vector $(1, \dots, 1)$
 1_A^S indicator function for $A \subseteq S$
 $\mathcal{A} = \{a_1, \dots, a_n\}$ assets
 C (price of a call)
 e_i the i th standard unit vector
 $\mathcal{E}_P(X)$: expected value of X with respect to probability P
 Φ_i asset holding process
 K (strike price)
 μ_X : expected value of X
 $\Omega = \{\omega_1, \dots, \omega_m\}$ states of the economy
 P (price of a put)
 $\mathcal{P}_i = \{B_{i,1}, \dots, B_{i,m_i}\}$ state partition
 $\mathbb{P}$ probability
 r riskfree interest rate
 $\text{RV}(\Omega)$ vector space of all random variables from Ω to $\mathbb{R}$
 $\text{RV}^n(\Omega)$ vector space of all random vectors from Ω to $\mathbb{R}^n$
 $\rho_{X,Y}$: correlation coefficient of X and Y
 S (price of stock or other asset)
 $\sigma = (s_1, \dots, s_m)$ state vector
 σ_X^2 : variance of X
 $\sigma_{X,Y}$: covariance of X and Y
 Θ_i portfolio
 $\mathcal{V}_0$ initial cost function
 $\mathcal{V}_T$ payoff function

Greek Alphabet

A α alpha	H η eta	N ν nu	T τ tau
B β beta	Θ θ theta	Ξ ξ xi	Υ υ upsilon
Γ γ gamma	I ι iota	O $\omicron$ omicron	Φ ϕ phi
Δ δ delta	K κ kappa	Π π pi	X χ chi
E ϵ epsilon	Λ λ lambda	P ρ rho	Ψ ψ psi
Z ζ zeta	M μ mu	Σ σ sigma	Ω ω omega