Lecture Notes in Statistics

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A Road to Randomness in Physical Systems

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For my parents

Preface

There are many ways of introducing the concept of probability in classical, i.e. deterministic, physics. This work is concerned with one approach, known as *"the method of arbitrary functions."* It was put forward by Poincaré in 1896 and developed by Hopf in the 1930's. The idea is the following. There is always some uncertainty in our knowledge of both the initial conditions and the values of the physical constants that characterize the evolution of a physical system. A probability density may be used to describe this uncertainty. For many physical systems, dependence on the initial density washes away with time. In these cases, the system's position eventually converges to the same random variable, no matter what density is used to describe initial uncertainty.

Hopf's results for the method of arbitrary functions are derived and extended in a unified fashion in these lecture notes. They include his work on dissipative systems subject to weak frictional forces. Most prominent among the problems he considers is his carnival wheel example, which is the first case where a probability distribution cannot be guessed from symmetry or other plausibility considerations, but has to be derived combining the actual physics with the method of arbitrary functions. Examples due to other authors, such as Poincaré's law of small planets, Borel's billiards problem and Keller's coin tossing analysis are also studied using this framework. Finally, many new applications are presented. They include bouncing balls, physical systems described by small oscillations (such as a simple pendulum and a coupled harmonic oscillator) and integrable systems (such as a heavy symmetric top).

This work shows that, from a mathematical point of view, most applications of the method of arbitrary functions follow from the fact that the fractional part of the product of a real number t and an absolutely continuous random vector X converges to a distribution uniform on the unit hypercube as t tends to infinity. This motivates the study of the speed at which convergence takes place in order to determine the practical relevance of the method of arbitrary functions for specific examples. New results on convergence, and tractable upper bounds for the rate of convergence are derived and applied.

Chapter 1 gives a more detailed overview of the contents of this work. Mathematical preliminaries are considered in Chap. 2. Mathematical results are presented in Sect. 3.1 (for the one dimensional case) and Sect. 4.1 (for higher dimensions). These results are applied to various examples in Sects. 3.2 and 4.2. The sections with applications may be read independently from the ones containing the mathematical results.

Hopf's contribution to the method of arbitrary functions is studied in detail in Chap. 5. The concrete examples where he applied this method – including his carnival wheel and Buffon needle examples – are considered in Sects. 5.1, 5.2 and 5.3. The concept of *statistical regularity* which he introduced to formalize the idea of the unpredictability of a dynamical system – in the sense of the method of arbitrary functions – and its close connection to ergodic theory are discussed in Sects. 5.4 and 5.6. The relation he established between the physical and statistical independence of variables describing a dynamical system is discussed in Section 5.5. This chapter concludes by extending the concept of statistical regularity to that of *partial* statistical regularity, thereby encompassing all examples considered in this work.

Chapter 6 studies the behavior of the fractional part of the random vectors $A^k X$ and $e^{tA} X$ when the integer k and the real number t tend to infinity, where A denotes an n by n matrix and X an n-dimensional absolutely continuous random vector. Necessary and sufficient conditions for the existence of a limit random variable that does not depend on the density associated with X are derived. They involve number theoretic properties of the generalized eigenvectors of the matrix A.

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Cambridge, Massachusetts, 1992 *Eduardo Engel*

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