

Lecture Notes in Statistics

Edited by J. Berger, S. Fienberg, J. Gani,
K. Krickeberg, I. Olkin, and B. Singer

71

Eduardo M.R.A. Engel

A Road to Randomness
in Physical Systems



Springer-Verlag Berlin Heidelberg GmbH

Author

Eduardo M. R. A. Engel
Department of Economics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
and
Departamento de Ingeniería Industrial
Universidad de Chile
República 701, Santiago, Chile

Mathematical Subject Classification: 60A99, 60E05, 60F99, 58F07, 58F11

ISBN 978-0-387-97740-9

ISBN 978-1-4419-8684-9 (eBook)

DOI 10.1007/978-1-4419-8684-9

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1992

Originally published by Springer-Verlag Berlin Heidelberg New York in 1992

Typesetting: Camera ready by author

47/3140-543210 – Printed on acid-free paper

For my parents

Preface

There are many ways of introducing the concept of probability in classical, i.e. deterministic, physics. This work is concerned with one approach, known as “*the method of arbitrary functions*.” It was put forward by Poincaré in 1896 and developed by Hopf in the 1930’s. The idea is the following. There is always some uncertainty in our knowledge of both the initial conditions and the values of the physical constants that characterize the evolution of a physical system. A probability density may be used to describe this uncertainty. For many physical systems, dependence on the initial density washes away with time. In these cases, the system’s position eventually converges to the same random variable, no matter what density is used to describe initial uncertainty.

Hopf’s results for the method of arbitrary functions are derived and extended in a unified fashion in these lecture notes. They include his work on dissipative systems subject to weak frictional forces. Most prominent among the problems he considers is his carnival wheel example, which is the first case where a probability distribution cannot be guessed from symmetry or other plausibility considerations, but has to be derived combining the actual physics with the method of arbitrary functions. Examples due to other authors, such as Poincaré’s law of small planets, Borel’s billiards problem and Keller’s coin tossing analysis are also studied using this framework. Finally, many new applications are presented. They include bouncing balls, physical systems described by small oscillations (such as a simple pendulum and a coupled harmonic oscillator) and integrable systems (such as a heavy symmetric top).

This work shows that, from a mathematical point of view, most applications of the method of arbitrary functions follow from the fact that the fractional part of the product of a real number t and an absolutely continuous random vector X converges to a distribution uniform on the unit hypercube as t tends to infinity. This motivates the study of the speed at which convergence takes place in order to determine the practical relevance of the method of arbitrary functions for specific examples. New results on convergence, and tractable upper bounds for the rate of convergence are derived and applied.

Chapter 1 gives a more detailed overview of the contents of this work. Mathematical preliminaries are considered in Chap. 2. Mathematical results are presented in Sect. 3.1 (for the one dimensional case) and Sect. 4.1 (for higher dimensions). These results are applied to various examples in Sects. 3.2 and 4.2. The sections with applications may be read independently from the ones containing the mathematical results.

Hopf's contribution to the method of arbitrary functions is studied in detail in Chap. 5. The concrete examples where he applied this method – including his carnival wheel and Buffon needle examples – are considered in Sects. 5.1, 5.2 and 5.3. The concept of *statistical regularity* which he introduced to formalize the idea of the unpredictability of a dynamical system – in the sense of the method of arbitrary functions – and its close connection to ergodic theory are discussed in Sects. 5.4 and 5.6. The relation he established between the physical and statistical independence of variables describing a dynamical system is discussed in Section 5.5. This chapter concludes by extending the concept of statistical regularity to that of *partial* statistical regularity, thereby encompassing all examples considered in this work.

Chapter 6 studies the behavior of the fractional part of the random vectors $A^k X$ and $e^{tA} X$ when the integer k and the real number t tend to infinity, where A denotes an n by n matrix and X an n -dimensional absolutely continuous random vector. Necessary and sufficient conditions for the existence of a limit random variable that does not depend on the density associated with X are derived. They involve number theoretic properties of the generalized eigenvectors of the matrix A .

This work corresponds to the Ph.D. dissertation I wrote while studying Statistics at Stanford University. I thank Kevin Coakley, Bradley Efron, Joseph Keller, Joseph Marhoul, Iain Johnstone, David Siegmund and Charles Stein for insightful comments and suggestions. I am especially indebted to Persi Diaconis for suggesting this research topic and providing generous advice throughout the writing of this work.

Cambridge, Massachusetts, 1992

Eduardo Engel

Table of Contents

1	Introduction	1
1.1	The Simple Harmonic Oscillator	1
1.2	Philosophical Interpretations	4
1.3	Coupled Harmonic Oscillators	4
1.4	Mathematical Results	6
1.5	Calculating Rates of Convergence	7
1.6	Hopf's Approach	9
1.7	Physical and Statistical Independence	10
1.8	Statistical Regularity of a Dynamical System	11
1.9	More Applications	11
2	Preliminaries	12
2.1	Basic Notation	12
2.2	Weak-star Convergence	13
2.3	Variation Distance	16
2.4	Sup Distance	18
2.5	Some Concepts from Number Theory	19
3	One Dimensional Case	26
3.1	Mathematical Results	26
3.1.1	Weak-star Convergence	26
3.1.2	Bounds on the Rate of Convergence	29
3.1.3	Exact Rates of Convergence	38
3.1.4	Fastest Rate of Convergence	42
3.2	Applications	42
3.2.1	A Bouncing Ball	42
3.2.2	Coin Tossing	44
3.2.3	Throwing a Dart at a Wall	48
3.2.4	Poincaré's Roulette Argument	50
3.2.5	Poincaré's Law of Small Planets	51
3.2.6	An Example from the Dynamical Systems Literature	52

4	Higher Dimensions	55
4.1	Mathematical Results	55
4.1.1	Weak-star Convergence	55
4.1.2	Bounds on the Rate of Convergence	59
4.1.3	Exact Rates of Convergence	69
4.2	Applications	71
4.2.1	Lagrange's Top and Integrable Systems	71
4.2.2	Coupled Harmonic Oscillators	76
4.2.3	Billiards	80
4.2.4	Gas Molecules in a Room	85
4.2.5	Random Number Generators	86
4.2.6	Repeated Observations	87
5	Hopf's Approach	89
5.1	Force as a Function of Only Velocity: One Dimensional case	91
5.2	Force as a Function of Only Velocity: Higher Dimensions	99
5.3	The Force also Depends on the Position	103
5.4	Statistical Regularity of a Dynamical System	107
5.5	Physical and Statistical Independence	115
5.6	The Method of Arbitrary Functions and Ergodic Theory	117
5.7	Partial Statistical Regularity	122
6	Non Diagonal Case	125
6.1	Mathematical Results	125
6.1.1	Convergence in the Variation Distance	125
6.1.2	Weak-star Convergence	129
6.1.3	Rates of Convergence	141
6.2	Linear Differential Equations	144
6.3	Automorphisms of the n -dimensional Torus	148
	References	151
	Index	153

List of Figures

1.1	Block and spring	1
1.2	Density of X and tX	3
1.3	Density of arcsine law	3
1.4	Coupled harmonic oscillators	5
1.5	Simple pendulum	8
1.6	Carnival wheel	9
3.1	Density that has the tent function as its characteristic function	42
3.2	Tent function	42
3.3	Bouncing ball	43
3.4	Limiting density for bouncing ball	44
3.5	Variables describing a coin's position	45
3.6	Partition of phase space induced by heads and tails	46
3.7	Wall painted with stripes	49
3.8	Relation between stripes and angle of release	49
3.9	Black and white sections on roulette wheel	51
3.10	Tent function	53
3.11	Saw-tooth function	53
4.1	Lagrangian top	71
4.2	Painted top	74
4.3	Coupled harmonic oscillators	77
4.4	System consisting of n coupled oscillators	78
4.5	Trajectory of particle	81
4.6	Original and auxiliary trajectories	82
4.7	Billiard table showing 45° degree angle	83
4.8	Rotated ellipse	84
5.1	Wheel and angle	91
5.2	Buffon's needle	100
5.3	Force component that depends on the wheel's position: symmetric case	104
5.4	Force component that depends on the wheel's position: asymmetric case	105
5.5	Graph of function leading to the counterexample	113
6.1	Drawing with parallel lines in unit square	131
6.2	Uniform distribution induced by H	131