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Harmonic Analysis of Operators on Hilbert Space

Second Edition

Springer

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(Deceased)

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Foreword

Sz.-Nagy and Foias had been planning for several years to issue an updated edition of their book *Harmonic Analysis of Operators on Hilbert Space* (North-Holland and Akadémiai Kiadó, Amsterdam–Budapest, 1970). This plan was not realized due to Sz.-Nagy's death in 1998. Sz.-Nagy's idea was to include all developments related to dilation theory and commutant lifting. Because there are several other volumes dedicated to some of these developments, we have decided to include in this volume only those subjects that are organically related to the original contents of the book. Thus, the study of $C_{1\cdot}$-contractions and their invariant subspaces in Chap. IX has its origins in Sec. VII.5, while the theory presented in Chap. X completes the study started in Secs. III.4 and IX.4 of the English edition.

The material in the English edition has been reorganized to some extent. The material in the original Chaps. I–VIII was mostly preserved, but the results in the original Chap. IX were dispersed throughout the book. We have added to several chapters a section titled *Further results*, where we discuss some developments related to the material of the corresponding chapter. The selection of topics was dictated by the authors' knowledge, and by space limitations. Many significant results are certainly omitted, and only some of these are listed in the bibliography. We apologize to those authors whose work did not receive proper mention.

Part of the work on this volume was performed during a semester visit by L. Kérchy to Texas A&M University. He wishes to express his gratitude to the Mathematics Department for its hospitality, and to acknowledge additional support from Hungarian research grant OTKA no. K75488. A first version of Chapters I–VIII was expertly typeset by Mrs. Robin Campbell. The authors extend their gratitude to her, as well as the Mathematics Department at Texas A&M University, for their support throughout this project. Jenő Hegedűs kindly translated the foreword to the Russian edition.

Béla Szőkefalvi-Nagy served as a mentor to all three authors. He influenced us through the clarity of his mathematical insight, and through his insistence that published results should answer the highest standards of originality, beauty, and exposition. We dedicate this edition to his memory.

College Station, Bloomington, and Szeged, June, 2009

C. Foias, H. Bercovici, and L. Kérchy

Foreword to the French Edition

In the theory of operators on Hilbert space, definitive results have long been known for self-adjoint, unitary, and normal operators—special types of operators, but types which are especially important in different branches of mathematics and theoretical physics. The theory of nonnormal operators, although also initiated a long time ago, using different methods, has not yet attained any such definitive form. The recent rapid progress in this field was stimulated largely by work of mathematicians in the USSR (M. G. Kreĭn, M. S. Livšic, M. S. Brodskiĭ, etc.) and in the United States (N. Wiener, H. Helson, D. Lowdenslager, P. Masani, etc.). The central concern of the first group was with characteristic functions of operators and the triangular models of operators obtained from them whereas the work of the second group was inspired primarily by prediction theory for stationary stochastic processes. But there is also a third research direction which started from the theorem on unitary dilations of contractions on Hilbert space (Sz.-Nagy, 1953) and was pursued by the authors of the present monograph and others (M. Schreiber, I. Halperin, H. Langer, W. Mlak, etc.). This last research direction has led, for instance, to an effective functional calculus for Hilbert space contractions. It also unifies, in a certain sense, the other two research directions. Thus the characteristic function of a contraction T appears in this study in an altogether natural way, namely by the "harmonic analysis" (or "Fourier analysis") of the unitary dilation of T, and this in turn was inspired by prediction theory.

The purpose of the present monograph is to give a detailed exposition of the information about a contraction that can be obtained from consideration of its unitary dilation.

Chapter I develops the fundamentals of the theory of isometric and unitary dilations, deriving these by several different methods. Most important are the dilations of semigroups with one generator, either discrete $(\{T^n\}, n = 0, 1, \ldots)$ or continuous $(\{T(s)\}, 0 \leq s < \infty)$. These are used throughout what follows. We also treat dilations of discrete commutative semigroups with several generators; here there are some beautiful and definitive results, but also some difficult unsolved problems. These results (Secs. 6 and 9) are not essential for the reader of the rest of the book.

In Chap. II we establish some geometric and spectral properties of the unitary dilation of a contraction T (or equivalently, of the discrete contraction semigroup $\{T^n\}$). Contractions are classified in terms of the asymptotic behavior of the powers of T and its adjoint T^*. The important notions of quasi-affinity and quasi-similarity are introduced. In Sec. 5 we prove the existence of an abundance of invariant subspaces for certain types of operators, a subject to which we return, with more powerful methods, in Chap. VII.

In Chaps. III and IV we develop a functional calculus for contractions T, based on applying spectral theory to the unitary dilation of T. The relevant functions are analytic on the unit disc, in particular the class of bounded analytic functions. A. Beurling's arithmetic of inner functions plays an essential role in connection with the "minimal functions" of contractions belonging to what we call the class C_0. Outer functions also play a key part in this calculus, especially in extending it to certain classes of analytic functions unbounded on the unit disc. Important applications are to continuous semigroups of contractions (considered as functions of their "cogenerators"), and to functions of accretive and dissipative operators, bounded or not (studied by use of their Cayley transforms). We define and analyze fractional powers of accretive operators, providing an illustration of the methods in a special case that has importance in its own right.

Chapter V, which is independent of the preceding chapters, sets forth the ideas and general theorems of the theory of operator-valued analytic functions. This material (except for Secs. 5 and 8) is used throughout the rest of the book. In particular, we establish the existence and properties of factorizations of these functions. Fundamental in this whole development are two lemmas (Sec. 3) on Fourier representations of Hilbert spaces and certain operators on them, with respect to bilateral or unilateral shifts on the spaces.

The characteristic function of a contraction T makes its appearance in Chap. VI, as the operator-valued analytic function corresponding to a certain orthogonal projection in the space of the unitary dilation of T, when this space is given its Fourier representation according to the lemmas of Chap. V. This yields at once a functional model for T. The functional model affords a tool for analyzing the structure of contractions and the relations among spectrum, minimal function, and characteristic function.

In Chap. VII we establish a one-to-one correspondence between the invariant subspaces of a contraction T and certain factorizations, called the "regular" factorizations, of the characteristic function of T. This correspondence allows us to demonstrate the existence and spectral properties of invariant subspaces for certain types of contractions (class C_{11}), thereby strengthening the results obtained by a more elementary method in Chap. II (Sec. 4).

Chapter VIII deals with contractions T that are "weak", that is, such that the spectrum of T is not the whole unit disc and $I - T^*T$ has finite trace. For these we find a variety of invariant subspaces that furnish a spectral decomposition, in much the same sense as in the theory of normal operators.

Chapter IX contains various further applications of the methods in the book: a criterion for a contraction to be similar to a unitary operator; relations of quasi-similarity for unicellular contractions; criteria for an operator to be unicellular; and finally, extension of these results, by use of a Cayley transformation, to accretive and dissipative operators and to continuous contraction semi-groups.

The Notes at the end of each chapter mention additional results, sketch the history of the subject, and give references to the literature.

The chapters are divided into sections and the sections into subsections. Results are designated as theorems, propositions, lemmas, and corollaries, and these are numbered separately within each section, as are the subsections and the formulas. The form of citations is the following: the second section in a chapter is called Sec. 2. Within that section, the third subsection is denoted by Sec. 2.3; the third formula by (2.3); the third theorem (or proposition, etc.) by 2.3. In references to other chapters, the appropriate roman numeral is prefixed; thus in referring to Chap. I we would write Sec. I.2.3, or (I.2.3), or Theorem I.2.3.

We have presupposed familiarity with the elements of the theory of Hilbert space (in particular with the spectral theory for unitary, self-adjoint, and normal operators). Indeed this monograph may be regarded as a sequel to the book *Leçons d'analyse fonctionnelle*[1] by F. Riesz and B. Sz.-Nagy and to the appendix added to it in 1955 by Sz.-Nagy.

An additional prerequisite is familiarity with the fundamental facts about the Hardy classes of analytic functions on the unit disc or a half-plane; these may be found in Hoffman's book [1]. We should mention also that Chaps. V and VI of our book have points of contact with the recent book by Helson [1]; but the two books overlap only slightly in the material covered.

Our thanks are due to our colleague István Kovács for his remarks offered in the course of reading the manuscript, and to the Publishing House of the Hungarian Academy of Sciences, and the Szeged Printing Shop for the care they showed in the technical preparation of this book.

Szeged and Bucharest, October 1966

Sz.-N.—F.

Foreword to the English Edition

Since this book was written in French three years ago, further progress has been made in several parts of the theory. We have made use of the opportunity of the translation into English to include some of the new results, and we have revised, improved, and completed many parts of the original.

We mention in particular the following changes. It was known (Theorem I.6.4) that every commuting pair of contractions has a (commuting) unitary dilation, but it was an open question whether this holds, without further restrictions, for commuting families of more than two contractions as well. Now we know by an example due to S. Parrott that the answer to this question is negative (Sec. I.6.3). (1) For the interesting subclass of power-bounded operators, consisting of the operators that admit ρ-unitary dilations, it is proved that all of them are similar to contractions (Sec. II.8). (2) A general dilation theorem is proved in Sec. II.2 for the commutants of contractions, and this theorem is applied later to the functional model of contractions of class C_{00} (Sec. VI.3.8). (3) The functional calculus for contractions is slightly extended so as to include certain meromorphic functions on the unit disc also (Sec. IV.1); this generalization is immediate, and proves to be natural and even necessary in the light of some recent research on the contractions of class $C_0(N)$; these are sketched in part 2 of the Notes to Chap. IX. (4) The important norm relation between the inverse of the characteristic function $\Theta_T(\lambda)$ of a contraction T and the resolvent of T, due to Gohberg and Kreĭn, is added as Proposition VI.4.2. (5) Factorizations of a simple example of contractive analytic function are studied in Sec. V.4.5, thus providing useful information in a problem raised by Theorem VII.6.2 (see the last part of the Notes to Chap. VII).

[1] References are to the English translation and are indicated by [*Funct. Anal.*].

There are still other places that underwent smaller or greater changes, and we benefited from a number of remarks made by colleagues, in particular by Ju. L. Šmuljan in Odessa, as well as by R. G. Douglas in Ann Arbor and Chandler Davis in Toronto, who kindly revised parts of the manuscript of the present English edition. Our sincere thanks are due to all of them.

Szeged and Bucharest, May 1969

Sz.-N.—F.

Foreword to the Russian Edition

The history of this book, whose Russian translation is recommended to the readers' attention, can be easily traced. In 1953, the famous Hungarian mathematician B. Szőkefalvi-Nagy published in the journal *Acta Scientiarum Mathematicarum (Szeged)* a theorem, now widely known, on the unitary dilation of contractions. This work was soon continued by the author and other researchers. In 1958, the young Romanian mathematician C. Foiaş joined in the elaboration of the theory of contractions. Since then a series of articles by B. Sz.-Nagy and C. Foias, under the common title *On the Contractions of Hilbert Space*, has appeared regularly in *Acta Szeged*.

This research has evolved into a well-developed theory, which plays an important role in modern functional analysis. We are glad to mention that this theory has numerous, sometimes unexpected connections with works of Soviet experts on operator theory. To begin with, B. Sz.-Nagy's original theorem was based on M. A. Naĭmark's result about generalized spectral functions. Later results of the authors of this book yielded explicit connections with the prediction theory of stationary processes, as well as with Beurling's theorem on the invariant subspaces of shifts. At first it seemed that these topics were far from the spectral theory of nonnormal operators developed by Soviet authors, even when they paid special attention to contractions.

The years 1963–1964 were very important in the theory of Hilbert space operators. During that time, B. Sz.-Nagy and C. Foiaş elaborated the functional calculus of contractions, and introduced the basic concept of the minimal function for a certain class of contractive operators. It was very impressive, and in our opinion quite unexpected, when in their work B. Sz.-Nagy and C. Foiaş arrived naturally at the concept of the characteristic function of a contraction, a concept that arose in the research of M. S. Livsič (in connection with operators close to unitaries). The characteristic function played a fundamental role in the research of many Soviet mathematicians for two decades. The authors of this book obtained an essentially new functional model for arbitrary contractive operators, and in this model the characteristic function appeared in a very explicit form. From this point on, the interaction between the research carried out by B. Sz.-Nagy and C. Foiaş, and that of the Soviet school of operator theory in Hilbert space, became clear. This interaction resulted in the solution of a series of hard and important problems in numerous chapters of the theory (operators similar to unitaries; unicellular contractions and dissipative operators; multiplication theorems for characteristic functions; methods connected with minimal functions; and others). For this reason it is not coincidental that the book contains many references to the works of Soviet mathematicians.

Another important event of the period around 1963 is connected with the success achieved by P. Lax and R. Phillips in the scattering theory of acoustic waves. These authors proposed an abstract scheme for scattering problems, and this led to a new interpretation of the *S*-matrix. Thus this concept, originally introduced in the quantum theory of scattering, has acquired a new life in classical mathematical physics. It turned out that the Lax–Phillips scheme is nothing else than a continuous analogue of the situation considered by B. Sz.-Nagy and C. Foiaş in their study of the special class of C_{00}-contractions. It became clear that the characteristic function of a contraction can also be regarded as the *S*-matrix of an appropriately formulated scattering problem.

We now witness the creation of a new important branch in the theory of Hilbert space operators. This involves a wide area of research including the theory of characteristic functions of various classes of operators, the calculus of triangular and multiplicative integrals, problems in the similarity theory of linear operators, several chapters of the theory of operators acting on spaces with an indefinite metric, certain aspects of the scattering theory of self-adjoint and non-self-adjoint operators, along with various applications to classical and quantum physics, and to constructive function theory. This research direction can hardly be presented within the framework of a sole monograph. Several books have appeared reflecting different facets of the aforementioned circle of problems. (Cf. for example, L. DE BRANGES [2], M. S. BRODSKIĬ [9], I. C. GOHBERG AND M. G. KREĬN [4], [7], P. D. LAX AND R. S. PHILLIPS [2], M. S. LIVSIČ [4], and H. HELSON [1].) A prominent place is now taken on this list by the monograph of B. Sz.-Nagy and C. Foiaş, summarizing their investigations. We are not sure that the title *Harmonic Analysis of Operators on Hilbert Space* fully reflects the content and the aims of the book, but it is in perfect harmony with the inner beauty of the theory, with its well-proportioned composition, and with its elegant style.

It is worth mentioning that the research topics discussed in the book are supplemented by historical comments and important remarks at the end of each chapter.

The book has been translated into Russian in close collaboration with the authors. Thanks to this cooperation, several small inaccuracies have been corrected, and numerous supplements have been inserted, bringing the contents of the present translation close to that of the English edition.

We have no doubt that the appearance of the Russian translation of this excellent book will be well received by researchers in functional analysis.

M. G. Kreĭn

Contents