Integral Geometry and Radon Transforms

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Springer

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ISBN 978-1-4419-6054-2 e-ISBN 978-1-4419-6055-9
DOI 10.1007/978-1-4419-6055-9
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010938299

Mathematics Subject Classification (2010): 53C65, 44A12

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

TO ARTIE

Skalat maðr rúnir rista
nema ráða vel kunni
þat verðr mörgum manni
es of myrkvan staf villisk.

Egils Saga Ch. 73, (ca. 1230).

Preface

This book deals with a special subject in the wide field of Geometric Analysis. The subject has its origins in results by Funk [1913] and Radon [1917] determining, respectively, a symmetric function on the two-sphere $\mathbf{S}^2$ from its great circle integrals and an integrable function on $\mathbf{R}^2$ from its straight line integrals. (See References.) The first of these is related to a geometric theorem of Minkowski [1911] (see Ch. III, §1).

While the above work of Funk and Radon lay dormant for a while, Fritz John revived the subject in important papers during the thirties and found significant applications to differential equations. More recent applications to X-ray technology and tomography have widened interest in the subject.

This book originated with lectures given at MIT in the Fall of 1966, based mostly on my papers during 1959–1965 on the Radon transform and its generalizations. The viewpoint of these generalizations is the following.

The set of points on $\mathbf{S}^2$ and the set of great circles on $\mathbf{S}^2$ are both acted on transitively by the group $\mathbf{O}(3)$. Similarly, the set of points in $\mathbf{R}^2$ and the set $\mathbf{P}^2$ of lines in $\mathbf{R}^2$ are both homogeneous spaces of the group $\mathbf{M}(2)$ of rigid motions of $\mathbf{R}^2$. This motivates our general Radon transform definition from [1965a] and [1966a], which forms the framework of Chapter II:

Given two homogeneous spaces $X = G/K$ and $\Xi = G/H$ of the same group G, two elements $x = gK$ and $\xi = \gamma H$ are said to be *incident* (denoted $x\#\xi$) if $gK \cap \gamma H \neq \emptyset$ (as subsets of G). We then define the *abstract Radon transform* $f \to \widehat{f}$ from $C_c(X)$ to $C(\Xi)$ and the *dual transform* $\varphi \to \check{\varphi}$ from $C_c(\Xi)$ to $C(X)$ by

$$\widehat{f}(\xi) = \int\limits_{x\#\xi} f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int\limits_{\xi\#x} \varphi(\xi)\, d\mu(\xi)$$

with canonical measures dm and $d\mu$. These geometrically dual operators $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ are also adjoint operators relative to the G-invariant measures dg_K, dg_H on G/K and G/H.

In the example $\mathbf{R}^2$, one takes $G = \mathbf{M}(2)$ and K the subgroup $\mathbf{O}(2)$ fixing the origin x_o and H the subgroup mapping a line ξ_o into itself. Thus we have

$$X = G/K = \mathbf{R}^2\,, \qquad \Xi = G/H = \mathbf{P}^2$$

and here it turns out $x \in X$ is incident to $\xi \in \Xi$ if and only if their distance equals the distance p between x_o and ξ_o. It is important not just to consider the case $p = 0$. Also the abstract definition does not require the members of Ξ to be subsets of X. Some natural questions arise for the operators $f \to \widehat{f}$, $\varphi \to \check{\varphi}$, namely:

(i) Injectivity

(ii) Inversion formulas

(iii) Ranges and kernels for specific function spaces on X and on Ξ

(iv) Support problems (does $\widehat{f}$ of compact support imply f of compact support?)

We investigate these problems for a variety of examples, mainly in Chapter II. Interesting analogies and differences appear. One such instance is when the classical Poisson integral for the unit disk turns out to be a certain Radon transform and offers wide ranging analogies with the X-ray transform in $\mathbf{R}^3$. See Table II.1 in Chapter II, §4.

In the abstract framework indicated above, a specific result for a single example automatically raises a host of conjectures.

The problems above are to a large extent solved for the X-ray transform and for the horocycle transform on Riemannian symmetric spaces. When G/K is a Euclidean space (respectively, a Riemannian symmetric space) and G/H the space of hyperplanes (respectively, the space of horocycles) the transform $f \to \widehat{f}$ has applications to certain differential equations. If L is a natural differential operator on G/K, the map $f \to \widehat{f}$ transfers it into a more manageable operator $\widehat{L}$ on G/H by the relation

$$(Lf)\widehat{}= \widehat{L}\widehat{f}\,.$$

Then the support theorem

$$\widehat{f} \;\; \text{compact support} \quad \Rightarrow f \;\; \text{compact support}$$

implies the existence theorem $L\mathcal{C}^\infty(G/K) = \mathcal{C}^\infty(G/K)$ for G-invariant differential operators L on G/K.

On the other hand, the applications of the original Radon transform on $\mathbf{R}^2$ to X-ray technology and tomography are based on the fact that for an unknown density f, X-ray attenuation measurements give $\widehat{f}$ directly and thus yield f itself via Radon's inversion formula. More precisely, let B be a planar convex body, $f(x)$ its density at the point x, and suppose a thin beam of X-rays is directed at B along a line ξ. Then, as observed by Cormack, the line integral $\widehat{f}(\xi)$ of f along ξ equals $\log(I_0/I)$ where I_0 and I, respectively, are the intensities of the beam before hitting B and after leaving B. Thus while f is at first unknown, the function $\widehat{f}$ (and thus f) is determined by the X-ray data. See Ch. I, §7,B. This work, initiated by Cormack and Hounsfield and earning them a Nobel Prize, has greatly increased interest in Radon transform theory. The support theorem brings in a certain refinement that the density $f(x)$ outside a convex set C can be determined by only using X-rays that do not enter C. See Ch. I, §7, B.

This book includes and recasts some material from my earlier book, "The Radon Transform", Birkhäuser (1999). It has a large number of new examples of Radon transforms, has an extended treatment of the Radon transform on constant curvature spaces, and contains full proofs for the antipodal Radon transform on compact two-point homogeneous spaces. The X-ray transform on symmetric spaces is treated in detail with explicit inversion formulas.

In order to make the book self-contained we have added three chapters at the end of the book. Chapter VII treats Fourier transforms and distributions, relying heavily on the concise treatment in Hörmander's books. We call particular attention to his profound Theorem 4.9, which in spite of its importance does not seem to have generally entered distribution theory books. We have found this result essential in our study [1994b] of the Radon transform on a symmetric space. Chapter VIII contains a short treatment of basic Lie group theory assuming only minimal familiarity with the concept of a manifold. Chapter IX is a short exposition of the basics of the theory of Cartan's symmetric spaces. Most chapters end with some Exercises and Further Results with explicit references.

Although the Bibliography is fairly extensive no completeness is attempted. In view of the rapid development of the subject the Bibliographical Notes can not be up to date. In these notes and in the text my books [1978] and [1984] and [1994b] are abbreviated to **DS** and **GGA** and **GSS**.

I am indebted to T.H. Danielsen, S. Jensen and J. Orloff for critical reading of parts of the manuscript, to R. Melrose and R. Seeley for suggestions, to F. Gonzalez, J. Hilgert, A. Kurusa, F. Rouvière and H. Schlichtkrull for concrete contributions mentioned at specific places in the text, and for various textual suggestions. Once more my sincere thanks to Jan Wetzel for skillful preparation of the manuscript and to Kaitlin Leach at Springer for her patient cooperation.

MIT — Sigurdur Helgason
May 2009

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