

# Springer Series in Statistics

*Advisors:*

P. Bickel, P. Diggle, S. Fienberg, K. Krickeberg,  
I. Olkin, N. Wermuth, S. Zeger

# Springer Series in Statistics

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(continued after index)

Peter J. Brockwell  
Richard A. Davis

# Time Series: Theory and Methods

Second Edition

With 124 Illustrations



Springer

Peter J. Brockwell  
Richard A. Davis  
Department of Statistics  
Colorado State University  
Fort Collins, CO 80523  
USA

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9

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To our families

# Preface to the Second Edition

This edition contains a large number of additions and corrections scattered throughout the text, including the incorporation of a new chapter on state-space models. The companion diskette for the IBM PC has expanded into the software package *ITSM: An Interactive Time Series Modelling Package for the PC*, which includes a manual and can be ordered from Springer-Verlag.\*

We are indebted to many readers who have used the book and programs and made suggestions for improvements. Unfortunately there is not enough space to acknowledge all who have contributed in this way; however, special mention must be made of our prize-winning fault-finders, Sid Resnick and F. Pukelsheim. Special mention should also be made of Anthony Brockwell, whose advice and support on computing matters was invaluable in the preparation of the new diskettes. We have been fortunate to work on the new edition in the excellent environments provided by the University of Melbourne and Colorado State University. We thank Duane Boes particularly for his support and encouragement throughout, and the Australian Research Council and National Science Foundation for their support of research related to the new material. We are also indebted to Springer-Verlag for their constant support and assistance in preparing the second edition.

Fort Collins, Colorado  
November, 1990

P.J. BROCKWELL  
R.A. DAVIS

\* *ITSM: An Interactive Time Series Modelling Package for the PC* by P.J. Brockwell and R.A. Davis. ISBN: 0-387-97482-2; 1991.

*Note added in the eighth printing:* The computer programs referred to in the text have now been superseded by the package ITSM2000, the student version of which accompanies our other text, *Introduction to Time Series and Forecasting*, also published by Springer-Verlag. Enquiries regarding purchase of the professional version of this package should be sent to [pjbrockwell@cs.com](mailto:pjbrockwell@cs.com).

# Preface to the First Edition

We have attempted in this book to give a systematic account of linear time series models and their application to the modelling and prediction of data collected sequentially in time. The aim is to provide specific techniques for handling data and at the same time to provide a thorough understanding of the mathematical basis for the techniques. Both time and frequency domain methods are discussed but the book is written in such a way that either approach could be emphasized. The book is intended to be a text for graduate students in statistics, mathematics, engineering, and the natural or social sciences. It has been used both at the M.S. level, emphasizing the more practical aspects of modelling, and at the Ph.D. level, where the detailed mathematical derivations of the deeper results can be included.

Distinctive features of the book are the extensive use of elementary Hilbert space methods and recursive prediction techniques based on innovations, use of the exact Gaussian likelihood and AIC for inference, a thorough treatment of the asymptotic behavior of the maximum likelihood estimators of the coefficients of univariate ARMA models, extensive illustrations of the techniques by means of numerical examples, and a large number of problems for the reader. The companion diskette contains programs written for the IBM PC, which can be used to apply the methods described in the text. Data sets can be found in the Appendix, and a more extensive collection (including most of those used for the examples in Chapters 1, 9, 10, 11 and 12) is on the diskette. Simulated ARMA series can easily be generated and filed using the program PEST. Valuable sources of additional time-series data are the collections of Makridakis et al. (1984) and Working Paper 109 (1984) of Scientific Computing Associates, DeKalb, Illinois.

Most of the material in the book is by now well-established in the time series literature and we have therefore not attempted to give credit for all the



results discussed. Our indebtedness to the authors of some of the well-known existing books on time series, in particular Anderson, Box and Jenkins, Fuller, Grenander and Rosenblatt, Hannan, Koopmans and Priestley will however be apparent. We were also fortunate to have access to notes on time series by W. Dunsmuir. To these and to the many other sources that have influenced our presentation of the subject we express our thanks.

Recursive techniques based on the Kalman filter and state-space representations of ARMA processes have played an important role in many recent developments in time series analysis. In particular the Gaussian likelihood of a time series can be expressed very simply in terms of the one-step linear predictors and their mean squared errors, both of which can be computed recursively using a Kalman filter. Instead of using a state-space representation for recursive prediction we utilize the innovations representation of an arbitrary Gaussian time series in order to compute best linear predictors and exact Gaussian likelihoods. This approach, developed by Rissanen and Barbosa, Kailath, Ansley and others, expresses the value of the series at time  $t$  in terms of the one-step prediction errors up to that time. This representation provides insight into the structure of the time series itself as well as leading to simple algorithms for simulation, prediction and likelihood calculation.

These algorithms are used in the parameter estimation program (PEST) found on the companion diskette. Given a data set of up to 2300 observations, the program can be used to find preliminary, least squares and maximum Gaussian likelihood estimators of the parameters of any prescribed ARIMA model for the data, and to predict future values. It can also be used to simulate values of an ARMA process and to compute and plot its theoretical autocovariance and spectral density functions. Data can be plotted, differenced, deseasonalized and detrended. The program will also plot the sample autocorrelation and partial autocorrelation functions of both the data itself and the residuals after model-fitting. The other time-series programs are SPEC, which computes spectral estimates for univariate or bivariate series based on the periodogram, and TRANS, which can be used either to compute and plot the sample cross-correlation function of two series, or to perform least squares estimation of the coefficients in a transfer function model relating the second series to the first (see Section 12.2). Also included on the diskette is a screen editing program (WORD6), which can be used to create arbitrary data files, and a collection of data files, some of which are analyzed in the book. Instructions for the use of these programs are contained in the file HELP on the diskette.

For a one-semester course on time-domain analysis and modelling at the M.S. level, we have used the following sections of the book:

1.1–1.6; 2.1–2.7; 3.1–3.5; 5.1–5.5; 7.1, 7.2; 8.1–8.9; 9.1–9.6

(with brief reference to Sections 4.2 and 4.4). The prerequisite for this course is a knowledge of probability and statistics at the level of the book *Introduction to the Theory of Statistics* by Mood, Graybill and Boes.

For a second semester, emphasizing frequency-domain analysis and multivariate series, we have used

4.1–4.4, 4.6–4.10; 10.1–10.7; 11.1–11.7; selections from Chap. 12.

At the M.S. level it has not been possible (or desirable) to go into the mathematical derivation of all the results used, particularly those in the starred sections, which require a stronger background in mathematical analysis and measure theory. Such a background is assumed in all of the starred sections and problems.

For Ph.D. students the book has been used as the basis for a more theoretical one-semester course covering the starred sections from Chapters 4 through 11 and parts of Chapter 12. The prerequisite for this course is a knowledge of measure-theoretic probability.

We are greatly indebted to E.J. Hannan, R.H. Jones, S.I. Resnick, S.Tavaré and D. Tjøstheim, whose comments on drafts of Chapters 1–8 led to substantial improvements. The book arose out of courses taught in the statistics department at Colorado State University and benefitted from the comments of many students. The development of the computer programs would not have been possible without the outstanding work of Joe Mandarino, the architect of the computer program PEST, and Anthony Brockwell, who contributed WORD6, graphics subroutines and general computing expertise. We are indebted also to the National Science Foundation for support for the research related to the book, and one of us (P.J.B.) to Kuwait University for providing an excellent environment in which to work on the early chapters. For permission to use the optimization program UNC22MIN we thank R. Schnabel of the University of Colorado computer science department. Finally we thank Pam Brockwell, whose contributions to the manuscript went far beyond those of typist, and the editors of Springer-Verlag, who showed great patience and cooperation in the final production of the book.

Fort Collins, Colorado  
October 1986

P.J. BROCKWELL  
R.A. DAVIS

# Contents

<b>Preface to the Second Edition</b>	vii
<b>Preface to the First Edition</b>	ix
<b>CHAPTER 1</b>	
<b>Stationary Time Series</b>	1
§1.1 Examples of Time Series	1
§1.2 Stochastic Processes	8
§1.3 Stationarity and Strict Stationarity	11
§1.4 The Estimation and Elimination of Trend and Seasonal Components	14
§1.5 The Autocovariance Function of a Stationary Process	25
§1.6 The Multivariate Normal Distribution	32
§1.7* Applications of Kolmogorov's Theorem	37
Problems	39
<b>CHAPTER 2</b>	
<b>Hilbert Spaces</b>	42
§2.1 Inner-Product Spaces and Their Properties	42
§2.2 Hilbert Spaces	46
§2.3 The Projection Theorem	48
§2.4 Orthonormal Sets	54
§2.5 Projection in $\mathbb{R}^n$	58
§2.6 Linear Regression and the General Linear Model	60
§2.7 Mean Square Convergence, Conditional Expectation and Best Linear Prediction in $L^2(\Omega, \mathcal{F}, P)$	62
§2.8 Fourier Series	65
§2.9 Hilbert Space Isomorphisms	67
§2.10* The Completeness of $L^2(\Omega, \mathcal{F}, P)$	68
§2.11* Complementary Results for Fourier Series	69
Problems	73

CHAPTER 3	
Stationary ARMA Processes	77
§3.1 Causal and Invertible ARMA Processes	77
§3.2 Moving Average Processes of Infinite Order	89
§3.3 Computing the Autocovariance Function of an ARMA( $p, q$ ) Process	91
§3.4 The Partial Autocorrelation Function	98
§3.5 The Autocovariance Generating Function	103
§3.6* Homogeneous Linear Difference Equations with Constant Coefficients	105
Problems	110
CHAPTER 4	
The Spectral Representation of a Stationary Process	114
§4.1 Complex-Valued Stationary Time Series	114
§4.2 The Spectral Distribution of a Linear Combination of Sinusoids	116
§4.3 Herglotz's Theorem	117
§4.4 Spectral Densities and ARMA Processes	122
§4.5* Circulants and Their Eigenvalues	133
§4.6* Orthogonal Increment Processes on $[-\pi, \pi]$	138
§4.7* Integration with Respect to an Orthogonal Increment Process	140
§4.8* The Spectral Representation	143
§4.9* Inversion Formulae	150
§4.10* Time-Invariant Linear Filters	152
§4.11* Properties of the Fourier Approximation $h_n$ to $I_{(v, \omega)}$	157
Problems	159
CHAPTER 5	
Prediction of Stationary Processes	166
§5.1 The Prediction Equations in the Time Domain	166
§5.2 Recursive Methods for Computing Best Linear Predictors	169
§5.3 Recursive Prediction of an ARMA( $p, q$ ) Process	175
§5.4 Prediction of a Stationary Gaussian Process; Prediction Bounds	182
§5.5 Prediction of a Causal Invertible ARMA Process in Terms of $X_j, -\infty < j \leq n$	182
§5.6* Prediction in the Frequency Domain	185
§5.7* The Wold Decomposition	187
§5.8* Kolmogorov's Formula	191
Problems	192
CHAPTER 6*	
Asymptotic Theory	198
§6.1 Convergence in Probability	198
§6.2 Convergence in $r^{\text{th}}$ Mean, $r > 0$	202
§6.3 Convergence in Distribution	204
§6.4 Central Limit Theorems and Related Results	209
Problems	215

## CHAPTER 7

Estimation of the Mean and the Autocovariance Function	218
§7.1 Estimation of $\mu$	218
§7.2 Estimation of $\gamma(\cdot)$ and $\rho(\cdot)$	220
§7.3* Derivation of the Asymptotic Distributions	225
Problems	236

## CHAPTER 8

Estimation for ARMA Models	238
§8.1 The Yule–Walker Equations and Parameter Estimation for Autoregressive Processes	239
§8.2 Preliminary Estimation for Autoregressive Processes Using the Durbin–Levinson Algorithm	241
§8.3 Preliminary Estimation for Moving Average Processes Using the Innovations Algorithm	245
§8.4 Preliminary Estimation for ARMA( $p, q$ ) Processes	250
§8.5 Remarks on Asymptotic Efficiency	253
§8.6 Recursive Calculation of the Likelihood of an Arbitrary Zero-Mean Gaussian Process	254
§8.7 Maximum Likelihood and Least Squares Estimation for ARMA Processes	256
§8.8 Asymptotic Properties of the Maximum Likelihood Estimators	258
§8.9 Confidence Intervals for the Parameters of a Causal Invertible ARMA Process	260
§8.10* Asymptotic Behavior of the Yule–Walker Estimates	262
§8.11* Asymptotic Normality of Parameter Estimators	265
Problems	269

## CHAPTER 9

Model Building and Forecasting with ARIMA Processes	273
§9.1 ARIMA Models for Non-Stationary Time Series	274
§9.2 Identification Techniques	284
§9.3 Order Selection	301
§9.4 Diagnostic Checking	306
§9.5 Forecasting ARIMA Models	314
§9.6 Seasonal ARIMA Models	320
Problems	326

## CHAPTER 10

Inference for the Spectrum of a Stationary Process	330
§10.1 The Periodogram	331
§10.2 Testing for the Presence of Hidden Periodicities	334
§10.3 Asymptotic Properties of the Periodogram	342
§10.4 Smoothing the Periodogram	350
§10.5 Confidence Intervals for the Spectrum	362
§10.6 Autoregressive, Maximum Entropy, Moving Average and Maximum Likelihood ARMA Spectral Estimators	365
§10.7 The Fast Fourier Transform (FFT) Algorithm	373

§10.8*	Derivation of the Asymptotic Behavior of the Maximum Likelihood and Least Squares Estimators of the Coefficients of an ARMA Process	375
	Problems	396
CHAPTER 11		
	Multivariate Time Series	401
§11.1	Second Order Properties of Multivariate Time Series	402
§11.2	Estimation of the Mean and Covariance Function	405
§11.3	Multivariate ARMA Processes	417
§11.4	Best Linear Predictors of Second Order Random Vectors	421
§11.5	Estimation for Multivariate ARMA Processes	430
§11.6	The Cross Spectrum	434
§11.7	Estimating the Cross Spectrum	443
§11.8*	The Spectral Representation of a Multivariate Stationary Time Series	454
	Problems	459
CHAPTER 12		
	State-Space Models and the Kalman Recursions	463
§12.1	State-Space Models	463
§12.2	The Kalman Recursions	474
§12.3	State-Space Models with Missing Observations	482
§12.4	Controllability and Observability	489
§12.5	Recursive Bayesian State Estimation	498
	Problems	501
CHAPTER 13		
	Further Topics	506
§13.1	Transfer Function Modelling	506
§13.2	Long Memory Processes	520
§13.3	Linear Processes with Infinite Variance	535
§13.4	Threshold Models	545
	Problems	552
	Appendix: Data Sets	555
	Bibliography	561
	Index	567