PERSPECTIVES ON MATHEMATICAL PRACTICES

LOGIC, EPISTEMOLOGY, AND THE UNITY OF SCIENCE

VOLUME 5

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal techniques like, for example, independence friendly logic, dialogical logics, multimodal logics, game theoretic semantics and linear logics, have the potential to cast new light no basic issues in the discussion of the unity of science.

This series provides a venue where philosophers and logicians can apply specific technical insights to fundamental philosophical problems. While the series is open to a wide variety of perspectives, including the study and analysis of argumentation and the critical discussion of the relationship between logic and the philosophy of science, the aim is to provide an integrated picture of the scientific enterprise in all its diversity.

Perspectives on Mathematical Practices

Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education

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Introduction

Is mathematics finally going through the Kuhnian revolution that the sciences or, more precisely, the philosophers, historians, sociologists, economists, psychologists of science, … have been able to deal with ever since the magical year of 1962? Apart from the fact that one cannot easily identify a book that has played the part that *The Structure* has played – of course, Lakatos' *Proofs and Refutations* comes pretty close, but it does not possess the generality of Kuhn's work – there seems to be plenty of reasons why mathematicians and philosophers of mathematics are reluctant to cheer the coming of a Kuhnian revolution in their favourite domain. Instead of a full-fledged historical-philosophical analysis (actually, many papers in this volume do precisely that, so it is quite unnecessary to duplicate their efforts in this introduction), let us just repeat once more the overused quote: "Mathematics is a free creation of the human spirit". In a nutshell it expresses the cherished beliefs that many share: mathematics stands on its own, free from any societal influence, individualist and immaterial, beyond space and time, in short, it occupies a universe of its own. This view usually, though not necessarily, goes together with a belief, if not a conviction, that mathematical capacities are innate, i.e., one is born a mathematician and a mathematical training merely serves to refine the powers already present. One just needs to remind oneself of the well-known story told by G. Hardy about his reluctance to familiarise the celebrated Indian mathematician Ramanujan with the notion of a proof in mathematics for fear of ruining his innate capabilities. Add to this that to a large extent the standard account of the life of Ramanujan is a romantic invention and we consider our point made (see Kanigel [1991] for a more 'realistic' biography of Ramanujan.)

Therefore, if it is your ambition, as it is ours, to set the Kuhnian revolution in mathematics on its tracks, what to do (to quote a famous political philosopher)? It seems obvious to us that the first thing to do is to look for a good description of the subject itself: *what kind of thing is this curious process we call mathematical practice*? The aim of this book is two-fold:

- first, to bring together a number of authors who have thought and are still thinking about what mathematical practice is in general as well as in detail, how it should be studied and how theories can be formulated and,
- secondly, to incorporate existing materials from other, though related disciplines, as is, e.g., the case for mathematical education. This is a well-developed research community with its own goals, methods, and theories, but somehow it does not seem to connect all that well with the philosophical community. We wish to show in this book that such connections are indeed possible, if not necessary (if only for thought-economical reasons: duplication is rarely a time-energy saving device).

The papers presented here can thus be subdivided into three major categories:

- a first set deals with the general theme of mathematical practice,
- a second set with specific themes that arise when one takes the viewpoint from a full-blooded description of mathematical practice and
- a third set, too important to classify under the second heading and already referred to above, namely the relation between mathematical practice on research level, academic and otherwise, and education.

General theme: how to deal with mathematical practice?

In this section four papers have been brought together that show quite different ways of approaching the question above, showing thereby that the issue of how one is to study mathematical practice is itself a very difficult and complicated problem. Rather than looking for a unifying framework, it is our belief that by presenting four rather disparate approaches we will hopefully succeed in convincing the reader that it is not very likely that such a unifying theory will be easily put together, if at all.

Consider the paper by Jody Azzouni, "How and Why Mathematics is Unique as a Social Practice". Azzouni recognizes the importance of mathematical practice as a subject worthy of philosophical reflection and he tries to identify the characteristics that distinguish, quite sharply in his mind, mathematical practice from other practices. His inspiration and arguments are drawn from mainly analytical philosophy, more specifically from philosophers such as Ludwig Wittgenstein, Michael Resnik, Saul Kripke, and Hilary Putnam. Azzouni aims to show, so we believe, that the same analytical tools that were used to deny the importance of mathematical practice can also be used to make a case in its favour.

Eduard Glas in "Mathematics as Objective Knowledge and as Human Practice" attacks the problem of the nature of mathematical practice from a different angle, namely philosophy of science, more precisely the work of Karl R. Popper and to be even more precise, the three world model Sir Karl was so fond of. We quote from Glas' paper to show the Popperian mode of thinking present: "Humankind has used descriptive and argumentative language to create a body of objective knowledge, stored in libraries and handed down from generation to generation, which enables us to profit from the trials and errors of our ancestors." It allows Glas to reach the same goal as Azzouni, i.e., to show that mathematics can be properly distinguished from other practices and to see no deep conflict between mathematics' objectivity and its being profoundly social.

A similar concern is present in the contribution of Robert Thomas, "The Comparison of Mathematics with Narrative". However, Thomas walks a different route. Combining elements of semiotic theory – Umberto Eco, Brian Rotman, and Hayden White are referred to –, and of philosophy of mathematics and science – here we find Stephan Körner, Hartry Field and Hilary Putnam as sources of inspiration –, he too aims to show that, although mathematical practice shares a number of similarities to stories and narratives, nevertheless it is at the same time quite distinct.

Finally a deeply social and to many disturbing sound can be heard in Sal Restivo's "Theory Of Mind, Social Science, and Mathematical Practice". Starting from a theory of mind that in its 'classical setting' is typically asocial, Restivo shows how it impregnates our standard view of mathematics. Socialising the mind leads to a socialisation of mathematics. His main inspiration is to be found in the work of one of the founding fathers of sociology, viz. Emile Durkheim and his seminal notion of 'practices as institutions'. It leads him to the conclusion that a number of 'old' questions have to be posed again because entirely new answers are in the making. To quote from his paper: "What are numbers (and what are all the basic concepts and processes that constitute mathematics?). What is a classroom? What are teachers and students? What is learning? What is truth? What does it mean to reason? What is a proof? The trick here is to see all of these old friends as *institutions.*"

As said above, these contributions show that a genuine theory of mathematical practice is possible. Genuine in the sense that it is not derived from any foundational theory such as formalism, logicism or intuitionism (and all varieties of constructivism that it generated), to name the three major schools of the twentieth century. Such a theory is also badly needed and the four different ways presented here of handling the subject show at the same time that much work remains to be done in terms of mutual comparisons and enrichments. This present situation however need not be an obstacle in order to have a closer look at more specific elements of mathematical practice. Indeed, they help to refine the general problems and they provide detailed accounts that can be usefully employed as test cases for the diverse general accounts. That is the justification for the next major part of this book.

Specific themes: taking mathematical practice seriously

What kind of (more) detailed problems should one expect? What seems rather obvious is that the same problems that came up during the development of the philosophy of science in the post-Kuhn era, have a fairly large chance of appearing within the (new kind of) philosophy of mathematics as well. For that reason it was to be indeed expected that someone should write something about incommensurability. After all, incommensurability – the problem of whether or not it is possible to compare different scientific theories – has been a core problem in the philosophy of science, so at least one should have a look at it from the mathematical perspective. This is precisely what Otávio Bueno in his paper "Incommensurability in Mathematics" tries to do. Rather surprisingly perhaps, Bueno presents a strong case *in favour of* incommensurability in mathematics. Unlikely as it seems – after all, is not a number a number whenever and wherever it appears, so should not comparability be guaranteed at all places and at all times? – he does a wonderful job, presenting specific case studies to back up his claim. In his own words: "Theory change in mathematics, just as theory change in science, becomes a more complex, more interesting and not a cumulative phenomenon. As with science, in mathematics sensitivity to meaning change is required. This means that a simple cumulative pattern of mathematical development doesn't seem to make sense of mathematics." The unavoidable conclusion does follow: revolutions in mathematics are possible.

Who speaks of incommensurability, unavoidably has at the back of his or her mind the problem of (mathematical) progress. As soon as some form of incommensurability, however weak, sneaks in, the problem of how to define progress poses itself. Madeline Muntersbjorn in her contribution "Mathematical Progress as Increased Scope" tries to deal with this difficult question. The suggestion she proposes is that "…, mathematicians are not like mapmakers who adhere to the environmentalist's ethic, 'take only memories—leave only footprints.' They are more like *terraformers*, science-fiction engineers who travel to inhospitable planets and struggle to make alien landscapes suitable for human settlement by adapting them to our perceptual needs and abilities via innovations in formal systems of signification.”. This to our minds powerful metaphor really begs to be further developed as it manages to steer a course between, on the one hand, the *Scylla* of strong forms of mathematical realism, including variations on Platonism, and, on the other hand, the *Charibdis* of relativism where “2 + 2 = ?” is a question on the same footing as “Did Sherlock Holmes have a homosexual affair with Watson?”.

The next step, of course, must be to provide case studies from these new perspectives. Of course, if we are thinking about mathematics of the past, then the easiest thing to do is to look at the history and historians of mathematics. Surely they must have thousands of case studies ready for use. Although we happily accept – how could one argue otherwise? – the work done by historians, nevertheless we do have something slightly different in mind. What we are talking about is a shift of focus, looking at the neglected or almost forgotten details, connecting elements that seem unrelated at first sight, so that, if successful, the philosophical relevance becomes clear(er).

A fine example of such an attempt is Brendan Larvor’s “Proof in C17 Algebra”. Although he treats the well-known mathematicians of that period – Girolamo Cardano, François Viète, Thomas Harriot, John Pell, and others –, he does look at it in a different way. He is interested in what one could describe as “proof styles”, making it possible to distinguish between a mathematical text of Viète in contrast with, say, Cardano. The connection with the philosophy of mathematics is easily made: what we count as proof today is something that took quite some time to develop and hence it is a delicate question to judge the *quality* of proofs. Hence, what are proofs? How can we distinguish proofs from arguments (if such a distinction is meaningful)? What will ‘future’ proofs look like (given that the proof concept is a mobile concept)?

Some of these questions, especially the question about proofs and arguments, are discussed in the contribution of Andrew Aberdein, “The Informal Logic of Mathematical Proof”. It is perhaps a bit surprising to see the names of Stephen Toulmin and Douglas Walton appear in a paper about mathematical proofs. After all, are these two authors not famously known for their work in argumentation theory and definitely not in proof theory? And what could be the involvement of argumentation theory in the understanding of mathematical proof? Aberdein’s claim is precisely that by looking at mathematics from an argumentation-theoretical point of view, aspects of the mathematical culture are brought into focus that, from a formal point of view, would be lost altogether. It thereby helps to refine our image(s) of mathematical practice. If proofs are situated elements of such practices, it would make much more sense to talk of *proof dialogues* as he proposes to do instead of proofs in an unqualified way, thereby inviting us to reify them.

There is however something extremely important that all contributions up to this point (mostly implicitly, sometimes explicitly) indicate: if mathematics is indeed a complex set of diverse mathematical practices, if indeed these practices are (also) shaped by social factors, dependent on societal circumstances, thus sensible to societal changes, and therefore very changeable, then to become a mathematician must also be constituted by a complex set of social processes. It cannot be a matter of 'simply' developing the faculties, capacities or powers already present in the genius' brain – "the seed is there, it merely needs to grow" –, rather it is a process that, to use a biological metaphor, aims at preparing an organism for a very specific environment. In short, any theory that takes itself seriously as a candidate for understanding mathematical practices, must deal with mathematics education. Hence the third part of this book.

The special case of mathematics education

As said above, the community of mathematics educators is a very well-established community of inquiry. However, it does not seem to connect very well with the philosophers of mathematics community (admitted that the latter group numberwise is rather small compared to the former one). Note that the intersection of the two sets is not empty: there are 'true' specialists, namely, philosophers of mathematics education. Nevertheless, it is our impression that they tend to be associated more strongly with the teachers than with the philosophers. In this third part of the book we wish to show that the two communities not only should, but really must meet more often and more intensely. In addition, it was our ambition to present the possible interactions between the philosophy of mathematics and mathematics education in as many ways as possible, from an abstract level to a very concrete level, from general considerations to case studies, from argumentation to narrative, from the institutional to the personal.

The first two papers of this part of the book are Leone Burton's "Mathematicians' narratives about mathematics – and their relationship to its learning" and Anthony and Dominic Peressini's "Philosophy of mathematics and mathematics education. The Confluence of Mathematics and Mathematical Activity". They both share the concern to show that mathematics and its philosophy on the one hand, and mathematics education and its philosophy on the other hand, have a lot to share. Do note that for both of them the idea to understand the proof concept as a social construct (we repeat, once again, without implying any form of deep relativism) is pivotal. For one thing, instead of the image whereby the ideal notion of logico-formal proof is transferred to the educational setting, we now have the image of a particular concept, viz. “proof”, arising in a particular community, usually referred to as “the” mathematicians, and then being transferred into a totally different setting, namely, the teaching context. Seen thus, there is little need for an *exact copy* of the proof concept in the classroom. Hence, all kinds of questions pop up: What kind of proof is required for pupils to get a ‘good’ feeling for mathematics? How do other arguments (here Aberdein’s paper is clearly relevant) function in the classroom? How do philosophical elements enter into the very same classroom?

It is more than obvious that answering these questions, apart from the philosophical setting, requires lots of case studies. We present here three such cases. We do know, of course, that there is wealth of materials available at the present moment. The emphasis, however, is not on a case study from the educational point of view, but from (at least) the philosophico-educational point of view. Jill Adler in her paper “Mathematical Practices in and across School Contexts” does precisely that. Analysing the situation in South Africa in the *post-apartheid* situation allows her to come to the conclusion that “… a decontextualised notion of mathematical practice makes no sense from the perspective of school mathematics, if at all. School mathematical practices are just that: practices dialectically produced by both mathematics and schooling.”

The second example concerns the Belgian, more specifically Flemish situation. As both editors of this volume are working in Belgium, it seems quite normal to have a case study “close to home”. However, as a case study, it is perhaps somewhat unusual and special because it involves a topic that is not often addressed, if at all, and equally often considered to be a borderline phenomenon. *In casu*, what we are talking about are journals for mathematics teachers, not to be confused with journals for educational mathematics, journals for philosophy of mathematics, journals for the philosophy of mathematics education, and so on. What is presented here in the paper by Ad Meskens, “The Importance of a Journal for Mathematics Teachers”, are mathematics teachers writing for mathematics teachers. What do they write about? What do they consider to be so interesting that their colleagues should know about it? It is important to note that the author is a mathematician himself, not a philosopher. So here we have at least one example of a person-in-the-field reporting from the field.

The third and last example, also the concluding piece of this book (and the editors of this volume are truly proud to be able to include a contribution from this author) is basically about a formula and its proof, viz. Reuben Hersh’s “On the Interdisciplinary Study of Mathematical Practice, with a Real Live Case Study”. Formulated thus, this seems at first sight hardly innovative, refreshing or stimulating. However, in the unique style that is Hersh's own, we are invited to walk along with him and, indeed, think about a formula and its proof, but in such a way that at every step the links to education, to philosophy, and, of course, to mathematics become clear, and, in fact, refreshing and stimulating. It is worthwhile to state here his conclusion: "This much can be said. Mathematics really exists. It is going on, it is taking place, it has been around a long time and is here to stay. If your vocabulary insists that it is not real, and since in any ordinary meaning of the word it is not "fictional", then you must find some other kind of ontology, neither "real" in your sense nor fictional in any sense, to place it in."

In this sense, we have made a full circle, starting from philosophical considerations to the most concrete case study imaginable, back to philosophy with a refreshed mind.

A concluding remark and thought

First the remark: a special feature of this volume is that all the authors present here did actually meet physically at an international conference, *Perspectives on Mathematical Practices* (*PMP2002*), held at *Vrije Universiteit Brussel*, between 24 and 26 October 2002. It was organized by the *Centre for Logic and Philosophy of Science* (*CLWF*) at the same university (for the full program, see http://www.vub.ac.be/CLWF/PMP2002). This explains for a part the coherence among the diverse contributions, presented here. The other part explaining the coherence has to do with the fact that a selection has been made out of all contributions at the conference. However, the remaining contributions are not lost to the reader for these have been published in the logico-analytical journal *Logique et Analyse* (Van Kerkhove & Van Bendegem [2002]). The table of contents of this special volume can be found at: http://www.vub.ac.be/CLWF/L&A/. Finally, we like to mention that a discussion and mailing group has been launched at http://www.vub.ac.be/CLWF/mathprac/, to ensure that the debates will indeed continue.

And then the thought: there is a famous quote attributed to the famous French mathematician Jean Dieudonné (see [1982], p. 23): "Celui qui m'expliquera pourquoi le milieu social des petites cours allemandes du XVIII^{e} siècle où vivait Gauss devait inévitablement le conduire à s'occuper de la construction du polygone régulier à 17 côtes, eh bien, je lui donnerai une médaille en chocolat." ("The person who will explain to me why the social setting of the small German courts of the 18th century wherein Gauss lived forced him inevitably to occupy himself with the construction of a 17-sided regular polygon, well, him I will give a chocolate medal.") Although perhaps this book is not a straightforward answer to Dieudonné's worry, at least it seems reasonable to start to think about what kind of chocolate and what kind of medal we would like to have.

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