Galois Connections and Applications

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Preface

Galois connections provide the order- or structure-preserving passage between two worlds of our imagination – and thus are inherent in human thinking wherever logical or mathematical reasoning about certain hierarchical structures is involved. Order-theoretically, a Galois connection is given simply by two opposite order-inverting (or order-preserving) maps whose composition yields two closure operations (or one closure and one kernel operation in the order-preserving case). Thus, the “hierarchies” in the two opposite worlds are reversed or transported when passing to the other world, and going forth and back becomes a stationary process when iterated. The advantage of such an “adjoint situation” is that information about objects and relationships in one of the two worlds may be used to gain new information about the other world, and *vice versa*. In classical Galois theory, for instance, properties of permutation groups are used to study field extensions. Or, in algebraic geometry, a good knowledge of polynomial rings gives insight into the structure of curves, surfaces and other algebraic varieties, and conversely. Moreover, restriction to the “Galois-closed” or “Galois-open” objects (the fixed points of the composite maps) leads to a precise “duality between two maximal subworlds”.

Though the name “Galois connections” refers to the classical example of Galois theory, where subfields of a field and subgroups of the automorphism group are related by a dual lattice isomorphism, a richer supply of examples comes from certain relations of annihilation or orthogonality, well-known from various branches of algebra, geometry and topology – and in fact, this was the starting point for Garrett Birkhoff when he invented in the thirties of the past century his general “polarities” as Galois connections between power sets, arising from relations between the ground sets. The more general order-theoretical type of Galois connections (*alias* Galois connexions or Galois correspondences) as defined above is due to Oystein Ore (about 1940).

One or two decades later, the even more general notion of adjoint functors was discovered by workers in category theory, and various very useful types of categorical Galois correspondences were subsequently invented.

Concerning the position of Galois connections within mathematics, it has to be admitted that deep theorems rarely are immediate consequences of the general theory of Galois connections, but usually require some extra tools and ideas stemming from the specific theory under consideration. But the general framework, supporting the intuition and suggesting the appropriate concepts, is established by the discovery of the “right” underlying Galois connection, followed by a good characterization of the “Galois-closed” (or “Galois-open”) elements or sets. And there is no doubt that many proofs become shorter, more elegant and more transparent in the language of Galois connections.

The aim of this book is to present not only the main ideas of general Galois theory, of concepts related to Galois connections, and of their appearance in various classical and modern areas of mathematics, but also to show how they can be applied in other disciplines. The book consists of 15 contributions, written by authors who are specialists in diverse fields of mathematics and who use Galois connections in their work. Although the majority of the contributions are dealing with themes of classical and universal algebra, the reader will also find many logical, order-theoretical, topological and categorical aspects – and, last but not least, several practical applications, sometimes even in extra-mathematical fields like linguistics, digital representation of graphics, or data analysis. In fact, the methods of modern formal concept analysis, based on polarities arising from certain relations between objects and attributes, have applications in quite diverse non-mathematical areas of sciences and arts. Readers interested in that theory, its mathematical background and its applications, are referred to the monograph *Formal Concept Analysis – Mathematical Foundations* by B. Ganter and R. Wille (Springer-Verlag, 1999).

In the first contribution, which is also aimed at “novices” in the theory of Galois connections, M. Ern presents the basic ideas and concepts: order-theoretical Galois connections in their original contravariant form with antitone (order-inverting) maps and in their covariant form with isotone (order-preserving) maps, also referred to as adjunc-

tions or “mixed” Galois connections. The classical examples come from polarities and axialities, i.e. Galois connections and adjunctions, respectively, between power sets. A journey through three centuries of mathematics illustrates the historical development of the main concepts in that area. After a discussion of diverse examples in classical Galois theory and other parts of algebra, the pioneer achievements of the twentieth century are sketched, in particular, Birkhoff’s idea of polarities and Ore’s Galois connexions. The journey ends with a survey of residuation theory, where binary operations having adjoint translations are the central tool. More recent developments, in particular, the categorical types of Galois correspondences and the even more general adjoint functors had to be omitted in this historical introduction, due to space limitations. Readers interested in these topics but not familiar with categorical thinking may consult, for example, the book by Adámek, Herrlich and Strecker: *Abstract and Concrete Categories* (J. Wiley & Sons, 1990).

In *Dyadic Mathematics – Abstractions from Logical Thought*, R. Wille sets out the philosophical, logical and mathematical ideas furnishing the foundations of formal concept analysis, and hence of concrete applications of Galois connections. He demonstrates how human logical reasoning is based on concepts and how these may be formalized mathematically by forming contexts and their concept lattices. As the title indicates, the emphasis is on dyadic conceptions and results in order and lattice theory, contextual logic, algebra, and geometry. This survey article is of interest not only for people looking for applications of Galois connections, but also for those who want to understand the logical and philosophical background.

Three other contributions are devoted to some order- and lattice-theoretical aspects of Galois connections and their applications in other fields.

In “*Galois Connections and Complete Sublattices*” by K. Denecke and S. L. Wismath, three different methods are described to characterize complete sublattices of a complete lattice, by using a conjugate pair of additive closure operators, by special closure or kernel operators defined on the given complete lattice, and by so-called Galois-closed subrelations. Basic tools here are contexts and concept lattices, known from formal concept analysis.

The frequently observed phenomenon that approximation properties of certain relations on ordered sets are closely related or even equivalent to certain (infinite) distributive laws for the given ordered structures or some of their completions, is studied in “*The Polarity between Approximation and Distribution*” by M. Ern. It turns out that the right framework for those investigations is a certain polarity coming from an elementary relation defined in terms of a fixed closure operation on a frame (*alias* locale, a pointfree abstraction of topology). Via that polarity, the Galois-closed approximating subsets are in one-to-one correspondence with the sublocales containing all closed elements, and the Galois-closed strongly approximating subsets correspond to the completely distributive sublocales. These results encompass many specific representation theorems in order theory and related fields such as domain theory and pointfree topology.

Adjoint pairs of maps on the integers or natural numbers are studied by J. Lambek in “*Iterated Galois Connections in Arithmetic and Linguistics*”. He points out that such adjoint pairs simply correspond to infinite subsets with an infinite complement, and that for maps that are unbounded on both sides, all iterated left and right adjoints exist. In the second part, some categorical generalizations lead to the notion of pregroups, which have interesting applications in a modern treatment of linguistic problems. In particular, they provide a new algebraic approach to the structure of sentences in natural languages.

Often it is necessary to define Galois connections for proper classes rather than sets. In order to circumvent set-theoretical difficulties in that extended setting, one usually assumes the existence of a “common universe”. One of the most important Galois connections of the class-theoretical type comes from the relation of satisfaction between algebras and equations, or more generally, between partial algebras and formulas. (The dual relation is the relation of validity.) This approach provides fundamental connections between algebraic structures and the equations (or formulas) they satisfy. The resulting Birkhoff-Tarski isomorphism theorems then establish dualities between varieties and equational theories, or between more general classes like quasivarieties and implicational theories, etc. Four papers here deal with these and some other polarities arising in universal algebra.

The survey paper “*Galois Connections for Partial Algebras*” by P. Burmeister deals with such polarities in the rather general context of many-sorted partial algebras, which are of still increasing importance not only in universal algebra but also in computer science. Since in that general realm there are several kinds of equations, one also has various corresponding relations of validity. Another polarity discussed in that paper comes from the relation between maps and formulas of a certain signature, saying that the map reflects the formula (or equivalently, preserves the negated formula). Here, the resulting polarity yields several interesting factorization systems for morphisms, known from category theory and universal algebra. In the last part of Burmeister’s paper, the power of attribute exploration via formal concept analysis is demonstrated for the discovery of interdependencies between properties of homomorphisms.

A standard algebraic approach to fragments of implicational or intuitionistic logic are Hilbert algebras and their deductive systems (containing the constant “true” and closed under *modus ponens*). In *Deductive Systems and Galois Connections*, I. Chajda and R. Halaš investigate a general algebra $\mathcal{A}$ with a constant 0 and its so-called *b-deductive systems*, where b is a binary term function of the algebra. These are closed under certain inference rules and form a closure system that is isomorphic, via a suitable axiomaticity, to the system of all congruences Θ satisfying $a\Theta c$ iff $b(a, c)\Theta 0$ and $b(c, a)\Theta 0$ (b -regular congruences for short). The basic tools for applications to congruence theory are so-called Fichtner terms of a variety, i.e. binary terms such that every congruence of any algebra in the variety is b -regular (it suffices to ensure that property for the identity relations). The b -deductive systems of any algebra in a variety having a Fichtner term turn out to be precisely the congruence kernels. In the second part, the polarity induced by the relation $R = \{(b, D) : D \text{ is a } b\text{-deductive system of } \mathcal{A}\}$ between binary terms and subsets of an algebra is investigated, and its Galois-closed sets are characterized for certain cases.

In “*Complexity of Terms and the Galois Connection Id – Mod*”, K. Denecke and S. L. Wismath study the influence of term complexity on the Galois connection between identities and algebras. Calling a variety k -normal when both sides of any of its nontrivial equations have complexity at least k , they extend several central results on nor-

mal varieties to the k -th level and illustrate the limitations of such extensions.

In the survey “*Q-Independence and Weak Automorphisms as Galois Connections*”, K. Głazek and S. Niwczyk present two further kinds of Galois connections appearing in universal algebra. These are related to general notions of independence and of weak automorphisms. In the first case, the basic relation for the required polarity is defined between subsets and partial maps of a fixed algebra: a subset X is related to a partial map p if the latter extends to a homomorphism on the subalgebra generated by X whenever X is the domain of p . In the second case, the basic relation is defined between permutations of a fixed set and clones over that set, and it holds iff the permutation preserves the clone (in a natural way).

Another fundamental type of Galois connection for universal algebra is a central theme in several papers. This is the Galois connection (Pol, Inv) arising from the invariance relation “ f preserves ϱ ” between operations of an algebra A and n -ary relations on A . In that case, one says that f is a polymorphism of ϱ , or that ϱ is invariant under f .

In *Galois Connections for Operations and Relations*, R. Pöschel gives a systematic overview of the manifold Galois connections obtained by restriction or extension of the connection (Pol, Inv) to specified or generalized ranges, respectively. The main theorems yield intrinsic characterizations of the Galois-closed sets and the corresponding closure operators on both sides. The general (not necessarily finite) case requires certain local closure operators providing interpolation, while on the relational side, one has to consider so-called relational clones, for example, various kinds of Krasner algebras or clones, corresponding (via Galois connection) to classes of endomorphisms and (weak) automorphisms (different from those considered by Głazek and Niwczyk). Moreover, the (Pol, Inv) approach provides nice solutions of simultaneous concrete characterization problems for subalgebra lattices, congruence lattices and automorphism groups. On the other hand, there are useful extensions of (Pol, Inv) to partial functions, multifunctions and heterogeneous functions (cf. the article by P. Burmeister).

In *Galois Connections and Polynomial Completeness*, K. Kaarli explores different generalizations and variations of primality and polyno-

mial completeness (meaning that all functions, or at least all relation-preserving functions, are polynomials). His approach makes systematic use of the Galois connection (Pol, Inv) and demonstrates how and why local interpolation plays such a crucial role in the infinite case. Thus, for example, a local term function is one that can be interpolated by a term function on every finite subset, and an algebra is called locally primal if every function (in several variables) is a local term function. Variations of that concept are (local) congruence primality, (local) endo- or automorphism primality etc. (cf. the survey article by R. Pöschel). Special attention is paid to varieties having a majority or near-unanimity term, to order affine complete lattices, to algebras with distributive lattice reduct, and to endoprimal algebras.

Á. Szendrei in her contribution *A Survey of Clones Closed Under Conjugation* studies important complete sublattices of the lattice of all clones of operations on a finite set, using a Galois connection that can be derived from (Pol, Inv) .

The last three papers are devoted to themes where categorical methods help to generalize and to improve classical results of topology and Galois theory.

W. Gähler shows in his survey *Galois Connections in Category Theory, Topology and Logic* how partially ordered monads (a categorical notion very close to adjoint situations) provide the right framework for a very general development of abstract categorical topology and convergence theory. The basic idea here is to suitably generalize the “classical” filter monad, which has proved basic for many questions of convergence theory in the past. But, as recent developments have shown, there is also a large overlap of such topological theories with non-classical logics and computer sciences (see, for example, the book by St. Vickers: *Topology via Logic* (Cambridge Tracts in Th. Comp. Sc. 5, Cambridge University Press, 1989). In the second part of this paper, various types of fuzzy filters in locales and quantales are studied more thoroughly, and their interdependencies are illustrated by several diagrams.

J. Šlapal’s note *A Galois Correspondence for Digital Topology* describes an adjoint situation between the category of (generalized) closure spaces in the sense of Čech (where only extensivity and preser-

vation of inclusion and of emptiness is required) and the category of relational systems of a given (possibly infinite) arity. This extends the well-known binary case, leading to a concrete functorial isomorphism between partially ordered sets and Alexandroff (discrete) spaces. However, the general case is much more involved. For the objects of the resulting coreflective subcategory of closure spaces, connectedness coincides with a certain kind of “discrete path-connectedness”. Therefore, the study of such objects is helpful in digital topology, where these notions of connectedness play a crucial role.

In the final paper, G. Janelidze gives a survey of *Recent Developments in Categorical Galois Theory*. This rather advanced paper requires some familiarity with categorical methods and concepts. For readers who are not experts in that area, some preparatory literature is recommended, for example the book *Galois Theories* by F. Borceux and G. Janelidze (Cambridge Studies in Adv. Math. 72, Cambridge University Press, 2001), where the development from classical Galois theory to the modern categorical versions is described more comprehensively. In the present article, one subsection deals with categorical versions of A. R. Magid’s Galois theory for rings, and another is concerned with central extensions of universal algebras. But, a bit surprisingly, the theory begins with some purely topological considerations on categorical characterizations of connectedness and components. A Galois structure is then defined as a triple consisting of an adjoint pair of functors and two subclasses of objects in the involved categories, called fibrations, having suitable closure properties with respect to certain categorical constructions. Many examples are given, but most of them also require some categorical background. The fundamental theorem of Galois theory (in a categorical form) is based on Grothendieck’s formulation saying that the opposite category of subextensions of the field extension E/K is equivalent to the category of transitive $\text{Gal}(E/K)$ -sets. The chosen approach confirms that categorical language provides many improvements and simplifications of classical results: besides various related phenomena, it explains why the Galois theory of fields and the theory of covering spaces are just identical, if interpreted the right way.

The plan for a book on Galois connections originated with the participants of a conference on that subject held at the University of Potsdam

in 2001. The editors hope that this volume, though certainly not comprehensive enough to cover all the important themes in this area, will help to make an easy but beautiful concept more popular. Thus, if the value of Galois connections as a unifying concept in quite diverse branches of mathematics, but also its usefulness as a practical tool both in mathematics and other areas, become more evident by the contributions in this book, our main intention will have been fulfilled.

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