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MATHEMATICAL ECONOMICS

R. G. D. ALLEN

Second Edition

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and

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PREFACE

THIS book has evolved from three strands of thought. When I was first interested in mathematical economics in the early 1930's, it seemed to me that the main need was for a grounding in calculus and I wrote my textbook, *Mathematical Analysis for Economists* (1938), for that purpose. I did not venture into the higher reaches of algebra and I made no use of the complex variable. I left such matters to be written up by those with specialist applications to economics in mind. Subsequently the uses of matrix algebra, of vectors and complex variables, of operational processes and of other such mathematical devices have been greatly developed in many parts of mathematical economics. There are in my view no mathematical texts on higher algebra and on operational methods which are really suitable for economists.

A second development of the last twenty years is the growth of econometrics. This has been so rapid that I think there is some risk that the necessary development of economic theory, formulated in a way which makes econometric sense to a statistician, will lag behind rather seriously. Such formulations of economic theory must be in mathematical terms but simplified as far as possible.

Finally, the change in the direction of economic thought over the past twenty years has involved a considerable upheaval in the structure of economic theory. This is partly, though by no means entirely, the result of the work of Keynes. I believe that there is now a real need for some synthesis of the "new" economics, for some calm survey of the form and scope of economic theory.

With these things in mind, I feel that the best contribution I can make is not to extend my 1938 book but rather to write a completely different one, a text on economic theory written in mathematical terms. The present work is not mathematics for the economist, nor is it econometrics. It aims at a fairly systematic treatment of some of the more important and simpler parts of mathematical economics. My main problem is one of selection from a vast range of economic topics, of making the book reasonably up-to-date and of keeping it down to reasonable dimensions. Even if I include only what is of particular interest to me personally, I would still find I have too much. To those who look for other topics or different methods of treatment, I can only make the familiar excuses of the anthologist.

I am sure that the book is very different from what it would have been if I had written it five years ago. How long it will remain even approximately up-to-date I cannot pretend to guess. I feel, however, that many of the techniques I attempt to describe will be relevant for some time to come ; they are now relatively new and they need to be consolidated and absorbed into the general body of economic theory. Moreover, my guiding principle is to start with, and to remain as close as possible to, economic problems of the real world, simplified as an economist might simplify them, and translated into mathematics of no more than moderate difficulty. I hope that this approach will appeal to many economists now and in the future.

I could not write a book of this kind without assistance on a scale far beyond my powers to acknowledge. I am grateful for the constant encouragement and advice of my colleagues, particularly Professor Lionel Robbins, Professor James Meade, Mr. David Knox, and Mr. Ralph Turvey. Mr. W. M. Gorman and Dr. F. H. Hahn of the University of Birmingham have kindly read through the manuscript ; they see the need for a book on the present lines, they are ideally qualified to write it, and yet they stand aside and allow me to monopolise the market. Above all, I am in debt to Dr. Helen Makower, Dr. G. Morton and Dr. A. W. Phillips ; this is clear to anyone who reads the middle and later chapters as now written.

R. G. D. ALLEN

LONDON SCHOOL OF ECONOMICS

PREFACE TO SECOND EDITION

I am glad to have this opportunity of making some changes in the text. Firstly, I have corrected errors and misprints which escaped me, all too frequently, during my original proof-reading. Secondly, I have made numerous revisions in the wording and exposition throughout the text. I hope that these will make the argument more clear and precise.

Thirdly, I have included some additional references, mainly to books and articles published since 1956. It happens that half-a-dozen books of outstanding merit have appeared in the two years 1957 and 1958. They are Goldberg: *Introduction to Difference Equations* (1958); Murdoch: *Linear Algebra for Undergraduates* (1957); Kemeny, Snell and Thompson: *Introduction to Finite Mathematics* (1957); Thrall and Tornheim: *Vector Spaces and Matrices* (1957); Luce and Raiffa: *Games and Decisions* (1957); and Dorfman, Samuelson and Solow: *Linear Programming and Economic Analysis* (1958). None of these can be described, mathematically, as very simple; they vary in level from the moderately difficult to the quite advanced. But, equally, none of them can be ignored in the education of the mathematical economist.

Finally, I have completely re-written some critical sections: 1.9 and 5.8 on time lags in dynamic models; 2.3, 10.3, 17.1 and 17.9 on general economic equilibrium; and 16.2 on the dual problem in linear programming. Substantial parts of the chapters on vectors and matrices have been drastically re-cast. I have extended Appendix A which deals with the "practical" mathematics of operators and linear systems. I have added a completely new Appendix B on "modern" algebra to serve as an introduction to the rigorous and postulational development of algebra. In a sense, this is a complement to Appendix A. The strict, axiomatic approach of the modern algebraist should be of considerable interest in itself to the economic theorist concerned with model-building. In any case, I hope that the new Appendix provides an under-pinning for the algebra of the text, where a compromise is adopted in the interests of simplicity.

My thanks are due to a number of correspondents. I am particularly grateful for the suggestions made by Sven Danø of the University of Copenhagen; by Lucien Foldes of the London School of Economics;

PREFACE

by Maurice McManus of the University of Birmingham and the School of Business Administration, Minnesota ; by Peter Newman of the University College of the West Indies ; and by Ciro Tognetti of Centro per la Ricorca Economica ed Econometrica, Genoa.

R. G. D. ALLEN

UNIVERSITY OF CALIFORNIA, BERKELEY
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INTRODUCTION

WHETHER mathematical techniques can be, or should be, used in economics is a much-discussed question. The proof of the pudding is in the eating. The economist must be left to determine for himself, when he reads the following chapters, whether the exposition of some important economic theories in mathematical terms assists him in appreciating the theories and in working out their implications.

The object of the text is to give a summary, and to some extent a synthesis, of what mathematical economists have written on certain economic theories. It is introductory, but it is not elementary. It is directed to the economist rather than to the mathematician, but it assumes some little mathematical background.

The order in which the economic material is developed may seem unusual ; it is not the traditional approach. The aim is to treat in mathematical form those economic problems which students of economics are required to handle and which have some bearing on the facts of economic life. It is also to start with something quite simple and to proceed later to the more difficult. The simplicity of what can be loosely described as macro-dynamic economics, the starting point here, lies in the fact that only a few broad aggregates are considered ; the relation to real problems appears through the dynamic approach. The problems of decision-taking, by the consumer, the firm, or the " economic planner ", involve whole sets of variables and they are best reserved for later treatment.

Nor is the order that of increasing complexity of the mathematics. This is an economic, not a mathematical, text-book. The nature of the mathematical techniques used is rarely allowed to dictate the order of development. The kind of mathematics set out in Allen (1938) and Tintner (1954) is assumed throughout, that is a good deal of calculus and some algebra and geometry. The main concession is that time out is taken at convenient places to introduce more advanced mathematical techniques, at the cost of disturbing the flow of the economic development.

Further, this is a book on economic theory, not on applied economics or econometrics. It is inevitable that the development verges at times on the econometric, since the concentration is always on economic theories with some relation to actual problems. Some attempt is made to frame economic theories in such a way that they can be tested against empirical

data, though by no means always sufficiently for the statistician to go to work. Indeed, some of the economic analyses presented here were originally designed by their authors for immediate econometric application ; this is the case for the work of Leontief (1951, 1953) and Koopmans (1951), and perhaps also for the trade cycle theories of Hicks (1950). But the models treated here are completely *deterministic* ; econometric work generally requires that *stochastic* elements are included in the models.

Within the general field so demarcated, the method of treatment is straight-forward enough. The elements of macro-dynamic economic theory (Chapters 1, 2 and 3) show up the need for using differential and difference equations and for the description of oscillatory variation by means of complex variables and vectors (Chapters 4, 5 and 6). On this basis can be developed some fairly elaborate trade cycle theories, leading to the vital problems of economic regulation (Chapters 7, 8 and 9). General equilibrium analysis, both of the Walrasian and of the Leontief (input-output) types, is then taken up (Chapters 10 and 11). It is found that a good deal of vector and matrix algebra is needed (Chapters 12, 13 and 14) and that the theory of games is of relevance to economic problems (Chapter 15). The development of linear programming and decision-taking can then proceed (Chapters 16 and 17) with particular applications to the theory of the firm and of the consumer (Chapters 18 and 19). The concluding chapter takes up some problems in aggregation and the economics of welfare (Chapter 20).

It is evident in these chapters how much is drawn from the work of a limited number of economists of Anglo-American schools, from Hicks and Samuelson, Hansen and Harrod, Leontief and Koopmans, and (among the younger writers) from Barna, Baumol, Domar, Dorfman, Duesenberry, Goodwin, Klein, Makower, Morton, Phillips, Solow and Turvey. This is an expression of indebtedness, not of apology. For it is the work of these economists that is familiar to the present author ; and it is the work which will be in the minds of most readers of this text. Only the title of the book may be questioned, since these chapters are concerned, not so much with mathematical economics in general, but with certain economic theories of present interest, developed in mathematical terms by Anglo-American economists.

The chapters also give the impression of a very limited range of economic theories. There is little or nothing on expectations, or on international trade or sector analysis generally. Further reflection may show that the topics treated here, though limited in scope, are of central importance in the general body of economic doctrine. Other topics, dealing with expecta-

tions or international economics, can be developed from them. Moreover, the analyses both of the trade cycle and economic regulation (Chapters 7, 8 and 9), and of linear programming and decision-taking (Chapters 16, 17, 18 and 19), are largely a product of work done in the decade after the Second World War. They are coming to hold the centre of the stage in economic theory and all economists need to know at least what they are trying to achieve.

In conclusion, it must be stressed that mathematical economics is applied mathematics, a partnership between mathematics and economics. Results in mathematical economics of any interest can only be derived by an economist using mathematics. The same is true of other fields of applied mathematics such as engineering ; indeed, the economist can learn a good deal from the engineer both as regards the kind of mathematics to use and in the formulation of the technical problems.

There is a good deal of misunderstanding of the nature of mathematics and of its mode of application. Pure mathematics is sometimes described as a " language ", with the implication that it is easily translated into English. It is nothing of the kind ; rather it is a specialised form of logic, of reasoning. A mathematical proof may be quite incapable of " translation ", though the premises from which it starts and the consequences reached can be, and should be, put in " literary " form.

Further, the concentration in mathematical teaching on " proofs ", on " Q.E.D. " at the bottom of the page, tends to suggest that pure mathematics proves theories. It does nothing of the kind ; it simply argues from premises to conclusions and the premises may be any self-consistent set of axioms which anyone cares to propose. Theories only arise in a particular subject matter, whether economics or electrical engineering, and so in applied mathematics. The theories then involve clothing the premises with a certain " real life " garb and in interpreting the logical or mathematical consequences in the same way. It can be said, if no slip is made in the reasoning, that the consequences hold if the premises are valid. But this is not a proof of any theory in economics or any other field. Theories are to be tested against facts, either the premises or (more usually) the consequences. The testing of a theory can lead to its rejection as inconsistent with facts ; but it can never lead to the " proof " of the theory, but only to its provisional acceptance as not inconsistent with facts. Mathematical economics is best regarded, therefore, as the process of following up the consequences of a particular set of self-consistent axioms with economic content. The proof is the establishment of the consequences of the axioms, not of the validity of the theory.

If mathematics is no more than a form of logical reasoning, the question may be asked : why use mathematics, which few understand, instead of logic which is intelligible to all? It is only a matter of efficiency, as when a contractor decides to use mechanical earth-moving equipment rather than picks and shovels. It is often simpler to use pick and shovel, and always conceivable that they will do any job ; but equally the steam shovel is often the economic proposition. Mathematics is the steam shovel of logical argument ; it may or it may not be profitable to use it. The point is that economic facts are extraordinarily complicated so that the steam shovel of mathematics is to be expected to be the most efficient way of delving into them. To maximise the relation of theory to fact, to minimise the simplification away from reality, it is usually safer to operate in mathematical terms. An economist who ventures to set up a theoretical model of empirical content is well advised to do so in explicit mathematical form. He risks failure if he does not ; or, at least, he is liable to overlook some cases or possibilities which may be important, and to make empirical testing of his model more difficult.

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- Note* : In a reference to an article in a Journal, the figure in bold type represents the volume number, and the following figures the pages of the article.