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Introduction to Non-Linear Optimization

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Preface

There are many books on non-linear optimization, so why another? A large part of the answer lies with the tremendous rate of growth in knowledge about problem-solving methods in this subject in recent times. At the time of writing, it is the 21st anniversary of the publication of the seminal paper by Fletcher and Powell (1963) which first made widely accessible the innovations of Davidon (1959). This is a turning point in the history of non-linear optimization methods and can be taken as marking the start of the modern exponential growth in research, software development and applications, all feeding off one another. It was then possible for the first time to locate the minimum of a smooth non-linear function efficiently without knowledge of second derivatives and with relatively little risk of going astray in the process. All that was needed was a computer with a sufficiently large memory. In the following year, even this restriction was relaxed (with some reservations) in the work of Fletcher and Reeves (1964), another milestone on the way. The next few years saw work upon improving the reliability of Newton's method and the fuse was lit for the explosion that followed.

The solution of linear optimization problems had been substantially understood at this time for almost two decades. These methods had been widely applied and they still are. However, one is always looking for better models of the real world and increasingly these have tended to become non-linear. In this way motivation has constantly been provided for the development of improved methods. The 1970s saw software development becoming a science (or perhaps a branch of engineering?) in itself, and reliable implementations, by no means easily achieved, of the better methods gradually appeared in an orderly, well-documented manner as, for example, in the NAG library. This has increasingly removed the burden of programming from the user to the expert. The reader will have noticed that we are talking about non-linear optimization *methods*. The theory of non-linear optimization dealing with constrained and unconstrained maxima and minima is an important branch of mathematics that had been long understood, but which can still provide difficulties for those in the practice of optimization who are not primarily mathematicians. The emphasis of this book is upon methods.

What is the situation in 1984? The growth continues unabated. The published literature compares in quantity with a well-known encyclopaedia of many volumes. New methods appear regularly, as do significant refinements of existing methods and enhancements of our theoretical understanding. So what hope is there of writing a book that will not be out of date almost as soon as it appears? Well, fortunately a backbone of methods covering almost all of the subject, and sup-

ported by comprehensive theory and widely available reliable implementations, has come into focus. These methods will not readily be superseded in practice for some time yet and the potential user will benefit from their study. In support of the worthiness of such a study is the observation that the growth in non-linear optimization over the last two decades has been evolutionary rather than revolutionary.

This is a book primarily about the main core of quite generally applicable non-linear optimization methods. The reader will also notice that some of the methods discussed have, or should have, passed out of use, but will find that these methods have a more than historical influence on the subject. A considerable unity within the field has emerged over the last decade or so. The author has had to be selective in the material discussed, but this has been done with a mind to emphasise unity and generality of application. Some old favourites (largely obsolete) do not appear. On the other hand, some obsolete methods nicely complete the picture and are included for this reason, although a related effect is that they generally provide some motivation for the better methods. The theory of minima is covered informally and in general the level of mathematical rigour is not very high; arguments often suggest results rather than prove them. This is therefore a book for actual or potential practising optimizers who need to know how methods work, how to select methods for the job in hand and how to use the chosen method. However, it is a mathematical subject and the reader must expect to see a considerable amount of mathematical argument on these pages. On the whole, the level is that usually covered in mathematics courses in numerate disciplines like physical science, engineering and computational mathematics by the end of the second year at university. More advanced topics are discussed briefly where relevant in the text. The main emphasis is upon linear algebra, which permeates the subject, and to a lesser extent upon multivariable calculus. Many standard texts are available that cover this material. The author finds particularly useful for reference the books by Stewart (1973) and Lancaster (1969) (linear algebra) and by Apostol (1957) (calculus).

The author's interest in non-linear optimization stems from advisory support to users at Liverpool University, from various projects in connection with the NAG library and from teaching. This book developed out of notes for a course of twenty lectures given to final year honours students taking degrees mainly in some form of computational mathematics. The project has grown, and it is feasible to cover only rather less than half of the material now presented in such a course. It is hoped that the full text will satisfy many of the requirements of research workers and optimization users in universities and industry.

I am indebted to many for assistance of one sort or another in the preparation of this book. In particular I should like to mention Shirley Hancock (once known in the optimization world as Shirley Lill) who suggested I should write the book when she finally came to terms with the fact that she would never get round to it herself; Dr. Brian Sutcliffe, a great mind and inspiration from my student days, who read a large chunk of the manuscript and provided detailed comment and boundless enthusiasm when it was badly needed; Professor Jim Alty, boss and friend, who

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Liverpool, 1984

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