METALOGIC

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*An Introduction to the Metatheory
of Standard First Order Logic*

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To my mother and to the memory of my father,

Joseph Walter Hunter

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Preface

My main aim is to make accessible to readers without any specialist training in mathematics, and with only an elementary knowledge of modern logic, complete proofs of the fundamental metatheorems of standard (i.e. basically truth-functional) first order logic, including a complete proof of the undecidability of a system of first order predicate logic with identity.

Many elementary logic books stop just where the subject gets interesting. This book starts at that point and goes through the interesting parts, as far as and including a proof that

it is impossible to program a computer to give the right answer (and no wrong answer) to each question of the form ‘Is — a truth of pure logic?’

The book is intended for non-mathematicians, and concepts of mathematics and set theory are explained as they are needed.

The main contents are: Proofs of the consistency, completeness and decidability of a formal system of standard truth-functional propositional logic. The same for first order monadic predicate logic. Proofs of the consistency and completeness of a formal system of first order predicate logic. Proofs of the consistency, completeness and undecidability of a formal system of first order predicate logic with identity. A proof of the existence of a non-standard model of a formal system of arithmetic.

The reader will be assumed to have an elementary knowledge of truth-functional connectives, truth tables and quantifiers. For the reader with no knowledge of set theory, here are very brief explanations of some notations and ideas that will be taken for granted later on:

1. *The curly bracket notation for sets*

‘{Fido, Joe}’ means ‘The set whose sole members are Fido and Joe’. ‘{3, 2, 1, 3, 2}’ means ‘The set whose sole members are the numbers 3, 2, 1, 3, 2’ (and this last set is the same set as {1, 2, 3}, i.e. the set whose sole members are the numbers 1, 2 and 3).

2. The epsilon notation for set-membership

' $n \in X$ ' means ' n is a member of the set X '.

3. The criterion of identity for sets

A set A is the same set as a set B if and only if A and B have exactly the same members. *Nothing else matters* for set identity.

4. The empty set, $\emptyset$

By the criterion of identity for sets [(3) above], if A is a set with no members and B is a set with no members, then A is the same set as B ; so if there is a set with no members, there is just one such set. We shall assume that there is such a set.

Further introductory material on set theory can be found in, for example, chap. 9 of Suppes (1957) or chap. 1 of Fraenkel (1961).

The book deals only with (1) *standard* (i.e. basically truth-functional) logic, and (2) *axiomatic* systems.

(1) Standard first order logic, with its metatheory, is now a secure field of knowledge; it is not the whole of logic, but it is important, and it is a jumping-off point for most other developments in modern logic. There seemed to me to be no book that tried to make accessible to non-mathematicians complete proofs of the basic metatheory of standard logic: hence this one.

(2) Axiomless systems (so-called 'natural deduction systems') are nowadays getting more popular than axiomatic systems, for formal proofs of theorems *inside* a system are generally shorter and easier to find with a natural deduction system than with an axiomatic one. But I find that complete proofs of *metatheorems* (theorems *about* a system) are in general longer and more laborious for natural deduction systems than for axiomatic ones; so, since I am mainly concerned with proofs of theorems about systems and not much concerned with proofs inside systems, and since anything you can get with a natural deduction system you can get with an axiomatic one, I have deliberately concentrated on axiomatic systems.

I hope that the book will equip those who are not mathematical specialists not only to tackle more advanced works on standard logic, such as those of Kleene or Mendelson or Shoen-

field or Smullyan,¹ but to frame philosophically interesting systems of non-standard logic and to prove metatheorems about them. For I believe that the logician's most urgent tasks at present lie in the field of non-standard logic. There is, for example, a sense of 'if' that is crucial to many everyday arguments; it seems clear to me that 'if' in that sense is not a truth-functional connective; and so I think it a scandal that the sense has not yet been adequately caught in any interpreted formal system with an adequate metatheory. I commend the task to my readers, who may find help in Church (1956) and in Hughes and Cresswell (1968).

The books I have borrowed most from are Mendelson (1964) and Margaris (1967). I thank the following people for ideas, information, or criticisms: Ross Brady, John Crossley, John Derrick, Len Goddard, Jeff Graves, Geoffrey Keene, Martin Löb, Angus Macintyre, Timothy Potts, Rowan Rockingham Gill, Harold Simmons, Dick Smith and Bob Stoothoff. My predecessors and colleagues in the Department of Logic and Metaphysics in the University of St Andrews, and especially Professors J. N. Wright and L. Goddard, made it possible for me to concentrate on this part of logic, and without them the book would not exist.

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*St Andrews,
December 1969*

¹ For detailed references, see pp. 262 ff.