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Advanced Calculus

Phil Dyke

**Professor of Applied Mathematics
University of Plymouth, UK**

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To my wife Heather

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Preface

This Work Out is intended as a student guide to the applications of differential and integral calculus to vectors. Such topics usually form part of the later years of an engineering or (physical) applied science degree, or the second or later stages of the first year of a mathematics degree. Students in these categories will, I hope, benefit most from this book. For completeness, a chapter on vector algebra is also included even though this is not strictly calculus.

In the past few years there have been many changes in what is taught to the 17 to 19 age group who study mathematics. Most of these changes have been gradual; evolutionary rather than revolutionary. However, one trend that is universally acknowledged by those who teach is the diminution in the amount of time devoted to those parts of mathematics that depend upon the student being able to gain ability in algebraic manipulation. There have been many reasons given for this, but the major reason must be the change in both *how* mathematics is taught and *what* mathematics is taught to young children. The 13 to 16 year old in particular is no longer exposed to long sessions of drill examples in, for example, multiplying out brackets, factorising, simplifying or solving simultaneous and quadratic equations. There is so much more interesting mathematics to do: relating mathematics to the real world, modelling, introducing statistics and decision mathematics. The presence of sophisticated graphics calculators and algebraic manipulation packages also lends weight to those who are looking for an ally in their arguments to dispense with algebraic manipulation skills. While the existence of automatic ways to do algebra (and calculus too) is welcomed by the mathematics community, it is also recognised that there are dangers in making these tools available to the mathematically unsophisticated. Students who have always had access to automatic ways of doing algebra and calculus may lack the ability to carry out algebraic manipulation to the level required to do even quite simple mathematical problems. In particular, even the better students who possess a certain amount of mathematical talent can find the examples and exercises in this book trickier than they should because they lack the experience, and the speed that comes with that experience to undertake the necessary algebra to solve the problem. If a student gets stuck, or makes an error in the algebra, then the ability to ‘see through to the end’ of a problem can all but disappear. A pianist does not rely on a machine for the scales, he or she needs to know them before becoming competent. With competence comes confidence which is mandatory for a pianist and not far short of mandatory for a student of mathematics. In my experience, students get a great deal of satisfaction in carrying out manipulation accurately and efficiently, and being good at this side of mathematics is highly correlated to being good at other aspects such as having insight, knowing when to stop one approach and to try an alternative etc. Therefore in writing this book there is less emphasis on manipulation than in equivalent books written twenty or thirty years ago, and where it is unavoidable the student has the choice of doing the algebra by hand or using artificial aids. From the above, however, you will be able to deduce that the author prefers students to have at least some skill in algebraic manipulation and as you follow through examples you are encouraged to follow the mathematics carefully.

In this book, a knowledge of one-dimensional calculus has been assumed, although the first chapter provides a brief run through of the basic notions of limits, continuity and differentiability. It is *not* a rigorous approach, but more a recapitulation of some of the concepts and techniques required for the rest of the book. If any of the material in

Chapter 1 is too hard, then you need to study a first course on calculus before going any further. Much of what is in Chapter 1 is, however 'A'-level mathematics. In Chapter 2, the notion of a partial derivative is introduced, again from the point of view of a user of mathematics rather than someone who is overly concerned with proving all results rigorously. The results established in this chapter will be very useful for the rest of the book. Chapter 3 is devoted to finding extremes (stationary values, or 'maxima and minima') in two dimensions. Chapter 4 is a brief foray into the study of optimisation. The emphasis is on applying calculus as against the numerical aspects which can be found in texts on numerical methods. Vectors are introduced in Chapter 5. It has been assumed that students have no previous knowledge of vectors, although most will have at least a passing acquaintance with them which will be useful. Chapters 6 and 7 contain the application of the differential calculus to vectors and will be new or nearly new material for most students. The main application areas for vectors that change are in geometry, where there is a sub-branch called differential geometry devoted to describing curves and surfaces using vectors, and mechanics. Not very many mechanics examples feature here since there is a *Work Out Mechanics* devoted to the subject. Chapter 8 is a short introduction to line integrals which have applications in physics, especially electromagnetism (and mechanics). Chapter 9 is devoted to multiple integration which is the application of integration to areas and volumes. Applications here include finding areas, volumes, centres of mass and moments of inertia. Finally Chapters 10 and 11 deal with surface integrals and relations between line, surface and volume integrals. Again the applications here are in areas such as electromagnetism, finding areas of curved surfaces and mechanics.

In writing this, the author has been influenced by over ten years of teaching this subject to undergraduate mathematicians at the University of Plymouth and before that teaching engineering students. Most of the examples contained here have been tried out on students and have passed some sort of road test. It is a pleasure to acknowledge all of them for the feedback they have given over the years. I also thank all those who have provided examples. Finally a warm thanks to my family who did not have as much of my time as I would have liked through the production stages of this text.

PHIL DYKE
University of Plymouth
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