Finite Elements
A Gentle Introduction

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Finite Elements

A Gentle Introduction

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Preface

The method called ‘finite elements’ has been developed over the last thirty years into a popular technique for solving a number of significant problems in engineering and the physical sciences. The design office of an engineering firm or consultancy, whether for civil, mechanical or electrical engineering, will most probably have available software which can predict the performance of a proposed design while it is still on the drawing board. Displacements and stresses, heat or groundwater flow, magnetic or acoustic fields ... all now can be predicted with an accuracy which is usually acceptable.

The areas to which the method can be applied are ever widening; topics which today are non-standard and the subject of research in universities and research laboratories will be developed into tomorrow’s software. Besides being actually used (or these days possibly because of it) the method is a topic in university courses. In addition to the utilitarian aspect, it draws together physics, mathematics, computer displaying of information, and has an elegance in its structure which is intellectually satisfying.

Both authors are currently working away from their home environments in countries full of interest (as are most countries) and with some features of special significance. Zimbabwe attracts tourists to see the grandeur of Victoria Falls, to visit the ruins of Great Zimbabwe set beneath high guarding rocks, and to see the game in the national parks. Swansea is near the Gower peninsula, an area of outstanding natural beauty. Every country has these features of special importance which are shown to tourists to give a brief overall impression; but there is much more to a country than can be shown in this way, and those who live and work there develop their own areas of interest and enjoyment. If finite elements were a country, this book would be a tourist guide as it discusses only the main features of the method. Working through it is like being a tourist (though sadly not as much fun) because the emphasis is on the basic concepts; the subdivision into elements, the formulation of the element and global equations and the mathematical background of the method. The justification for adding one more to the long list of ‘Introduction to Finite Elements’-type of book lies in that here not many details are visited, but the basic ideas are treated carefully and worked through with numerical examples as far as possible. To continue with the tourist analogy, this book is not so much a coach trip as a walking tour with ‘exercises’ on the way to break the strain of absorbing other people’s ideas and to fill in some details.

If you want to research the method or look in detail at particular aspects, you will need to refer to one of the several full treatments such as:

O. C. Zienkiewicz and R. L. Taylor (1994), *The Finite Element Method, Volumes 1–2*, 4th edition, McGraw-Hill.

E. B. Becker, G. F. Carey and J. T. Oden (1981), *Finite Elements, Volumes 1–6*, Prentice Hall.

T. J. R. Hughes (1987), *The Finite Element Method*, Prentice Hall.

The book grew out of lectures given at the (now) University of Brighton for students of engineering, computer science, and combined science in which mathematics is a main component, and it is hoped that it may prove useful for similar courses. Chapter 1 stands rather on its own; it may provide a useful introduction for engineering students who have met frameworks elsewhere, or it may be omitted. The other chapters form a dependent sequence with the last, on the application to elasticity, containing the most involved mathematics.

It has been found that learning can be made more interesting and students realise the usefulness of the method if *workshops* in a computer laboratory are given concurrently with the teaching of the theory. These sessions of say 2 hours can, using suitable software, illustrate and reinforce the ideas of the chapters. To encourage others to try this, a pamphlet containing a set of workshops to provide ideas, together with more detailed solutions to the exercises, is available on request from the publisher. If other lecturers have similar workshop ideas, they would be gratefully received and included; please e-mail henwood@maths.uz.zw or j.bonet@swansea.ac.uk.

Finally, we should thank those who have contributed to the development of this book: our students for teaching us how to teach, with no small generosity of spirit; and our colleagues for discussions, in particular John Barnes of the University of Brighton and Richard Wood from the University of Wales Swansea.

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