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MACMILLAN COLLEGE WORK OUT SERIES

Mathematical Modelling Skills

Dilwyn Edwards and Michael Hamson

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Preface

The activity of solving problems by mathematical modelling is by its very nature a practical and creative process involving a number of stages, all of which demand a range of skills. Many of these skills are gradually gained through practice and experience, and any teaching course involving mathematical modelling will inevitably be centred on practical work with example models. Essential though this is, it does mean that students are left to pick up the necessary skills as and when needed, in a fairly random way. The motivation behind this book is to direct students towards building up these skills in a systematic way, through carefully constructed exercises. Experience has shown that the necessary skills are most effectively acquired by working on such exercises in addition to and *in isolation* from the main (and usually confusing) business of tackling an actual model. This book should therefore be a very useful companion to any introductory course in mathematical modelling, whether in a scientific context or business environment.

This is not a book that can be *read*; rather it is meant to be *used*. Its purpose is to provide practice and training in the skills of mathematical modelling by working through a collection of worked examples and further exercises. It is not just a collection of exercises, however; each chapter either isolates a particular skill which is relevant at some stage in the modelling process, or deals with a particular modelling concept. These skills and concepts cannot be simply learnt, they have to be used, and it is only by practice that the necessary expertise can be developed. This explains the large number of examples and exercises given. They are from a variety of subject areas and cover a wide range of difficulty, from GCSE level to first or even second year undergraduate level. Some of the problems in Chapter 9, for example, are quite advanced. Within each chapter the exercises are numbered very roughly in order of difficulty and answers are provided where appropriate. The temptation to look at the answers should be resisted until every effort has been made to solve the problem! Most of the problems are new, but a number of the topics arise from the pioneering work of many colleagues from the Open University and the former polytechnics, in the area of teaching mathematical modelling. Many of the skills are common to a number of areas so there is inevitably some overlap between the chapters.

Generally speaking it will be found convenient to take the chapters in numerical order, but readers should be warned that they are not of equal length; the idea is to move on when it is felt that an adequate level of competence has been reached. Chapter 9 is particularly long because of the importance of differential equations in modelling. By working through the examples and exercises readers will increase their confidence and strengthen their model-building skills. The effort put in will be amply repaid when real modelling problems have to be tackled.

Readers new to mathematical modelling should first read the Introduction, and all readers should bear in mind that mathematical modelling is a structured process involving a number of stages. *The complete modelling process requires the combined application of all the various skills and concepts covered in this book.* Chapters 1–4 concentrate on the development of basic modelling skills. Each of Chapters 5–11 isolates a particular modelling concept and provides exercises aimed at developing skills in the use of that concept. It is not necessary to take the chapters from 5 onwards in strict numerical order. In many places it will be found advantageous to make use of computer software for mathematical manipulations,

graph-plotting and the calculation of results. Spreadsheets will be found very useful for general purposes and especially for discrete models. Software packages such as DERIVE or MATHCAD will greatly help in investigating continuous models, while packages such as MATLAB or MATHEMATICA will be useful for both discrete and continuous models. For simple models using random numbers, spreadsheets or MATHEMATICA can be used, but more complex stochastic models are best tackled using specialised discrete-event simulation software.

We have tried to eliminate typing errors and other mistakes but the large number of examples makes it unlikely that we have been totally successful. The authors would be very grateful for notification of any errors discovered by readers. Finally let us point out that by no means all of the skills of modelling have been covered here. We have not included examples of the application of the complete modelling process because such examples can be found in other books on modelling. Also modelling in practice is not usually a solitary activity but is carried out by a group working together on the problem. This demands skills of communication and organisation. The end result is finally often presented orally as well as in the form of a written report, and there are many skills involved there. Some advice on these aspects can be found in *Guide to Mathematical Modelling* published by Macmillan Press.

Introduction to Modelling

What is Mathematical Modelling?

Books dealing with mathematical methods sometimes illustrate their discussions with examples intended to show real-life applications. A hypothetical example might be 'A farmer finds that when he uses x kg of fertiliser on each m^2 of soil, his crop yield is $x^3 + 6x - 2$ kg m^{-2} '. In real life the farmer (or even the farmer's mathematician friend) would never 'find' such a thing. This is not to say that there could *not* be a formula in terms of x which predicts the crop yield when x kg of fertiliser is used. There may well be such a formula, which we call a *model*, and it could be very useful in making predictions, but it could not be just plucked out of the air. Mathematical models are patiently constructed using a well-tried process and can be based either on data (on crop yields and fertiliser in this case) or on *assumptions* (in this case about how crop yield responds to fertiliser treatment) or usually a combination of both. We can define mathematical modelling as the activity of translating a real problem into mathematics for subsequent analysis. A mathematical model will be created and its solution will usually provide information useful in dealing with the original real problem.

What are Real Problems?

Real problems can come from many different sources and at various levels of difficulty, from working out new traffic light settings to sorting out the badminton club fixtures, and from deciding the distribution of milk to home decorating and DIY. Professional mathematical modellers exist in industry and commerce working in many different areas. Elsewhere there are lots of common situations at work, home or leisure where mathematics is needed to solve a particular problem. In all cases there is some translating to be done from the problem into mathematics to form a mathematical model.

What is the Key Feature in Mathematical Modelling?

The interest lies not so much in solving mathematical equations as being able to make the most effective translation from the original problem into mathematics, so that the resulting model is of some practical use in solving the real problem. It is usually possible to get hold of computer software which will solve the equations. The main issue therefore lies in understanding the problem and its subsequent conversion into a mathematical form.

How is Modelling Carried Out?

We must not assume that the activity of formulating our problem in mathematical terms is easy; in fact it is usually much harder to do this than it is to solve the resulting equations. The difficulty is that the original problem probably will not be presented in an immediate mathematical form. Worse, it may not be fully specified, since the presenter is not a mathematician or does not actually know the full specification anyway. Thus we are concerned in understanding and perhaps modifying the original problem. We may also be concerned with what use is made of our model solution afterwards.

One of the most important points to realise is that the activity of modelling is a *process* which involves a number of clearly identifiable stages. The usual way of representing this process is by means of a modelling flow diagram similar to the one shown below, as originally

introduced by the Open University. The process is seen to be a *cycle* which may need to be traversed a number of times before the results are satisfactory. The point of the flow diagram is that it gives us a framework to refer to and acts as a channel for our thoughts and ideas.

What is the Main Objective of This Book?

The main objective is to train the reader in the skills that are required in building good mathematical models. We are not concerned primarily with explaining mathematical techniques that will then be used in analysing and solving the model, as there are plenty of textbooks already on this. The point is to concentrate first on the specification and understanding of the problem and then the resulting formulation of a model – this often turns out to be the most rewarding part of the procedure. The learning of mathematical skills and techniques is of course essential to support the modelling activity, but we must realise that ‘learning to use mathematics is not the same as learning mathematics’.

Why Should This Book be Used if We Don’t Want to be Professional Mathematicians?

Tackling real situations using mathematics is a challenge at all levels. Just think of a range of problems which everyone encounters from time to time: buying a car, loan repayment, household heating bills, bulk purchase, running a disco, preparing and reading timetables and routes, painting the house, measuring physical fitness, playing sport, placing a bet, foreign travel, parachuting etc. All of these activities have a quantitative element; understanding their nature and then making some decision in each case will normally need a mathematical representation or model.

Why Should This Book be Used if We *do* Want to be Professional Mathematicians?

Through tackling real problems at the starter level we can build up the confidence to take on more and more advanced situations to be met on industrial training or in the first job. Alongside modelling skills will be our development of mathematical, statistical and computing techniques necessary in solving the model. Professional mathematicians need the wide experiences gained through building models in applying their mathematics and statistics. With the help of powerful computers, mathematical models are used to help solve problems in production and distribution, design engineering, insurance services, economic forecasting and so on.

Haven’t Most of the Real Industrial Problems been Solved by Now?

Computer modelling has developed in sophistication so that major issues such as weather forecasting and economic prediction are more reliable than, say twenty years ago, but there

is plenty more to be done in tuning and adjusting these models to incorporate more features. This can only be done through a thorough understanding of an existing model, how it was formulated and what assumptions were made. There are many new situations arising where mathematical models are used, for instance in analysing the spread of diseases, pollution measurement, stock control, marketing a new product and many other areas.

Why is This Book Different from Other Books on Modelling?

We have concentrated on the first two boxes in the flow diagram by highlighting all the difficulties encountered and the skills needed. This is not to detract from the subsequent model solution and validation, but often the part of the modelling process found to be most difficult is that of specifying and formulating a model from an original problem. This book provides a series of exercises designed to develop the skills necessary in the various stages of the modelling process. Please see the Bibliography for examples of texts in which further modelling case studies can be found.

How to Use This Book

Each chapter either isolates a particular skill (or set of skills) relevant at some stage in the modelling process, or concentrates on showing how a particular modelling idea can be used. The aim is for you to develop your expertise with that particular skill or concept in the following way:

1. An outline of the **background** introduces the ideas and explains why they are needed and how they may be used. A brief list of the relevant mathematical skills is also given.
2. A set of **worked examples** is presented, illustrating the theme and its various applications. These should be carefully read through.
3. Further **exercises** are provided for you to test and extend your own understanding of and ability to apply the main ideas introduced in that chapter.
4. **Answers** are given to the exercises. In some cases it should be realised they are only *sample* solutions where there may well be other equally valid (or better) answers.

The main point to remember is that this is a book for *working* with, not just reading, and the effort put into the work will, we feel sure, be justly rewarded.