ALICE IN NUMBERLAND

Alice in Numberland
A Students' Guide to the Enjoyment of
Higher Mathematics

John Baylis
Rod Haggarty

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To Daniel, Gwen, Siân and Sarah

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INTRODUCTION

Failure to read this can damage your understanding

'It saddens me that educated people don't even know that my subject exists.'

Paul R. Halmos

What is our purpose in writing this book? Quite simply, to help students of mathematics *enjoy* their work. It seems to us that some students are good at mathematics and just get on with it, others tolerate it for its promised usefulness, but few actively enjoy it except in the rather trivial sense of preferring success to frustration.

We have set out quite deliberately to write an entertaining book whose content is serious mathematics. The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point, which is why we suggest that you regard this introduction as compulsory reading! None of the light-hearted sections are there just for entertainment; they are all meant to illuminate important mathematics. Now that we have alerted you to this, perhaps you will forgive us if we occasionally 'explain the point of the joke'. If you don't need the explanation, so much the better.

What we do assume of our readers is that they already have some interest in and commitment to mathematics as either students or teachers, and that they have some feeling that its pursuit could be enjoyable. Our domain is pure mathematics at the general level of sixth-form and early-undergraduate work, and the path we intend to take through this domain is deliberately not direct. It is winding, it branches at many points and it is often self-intersecting. It would be possible to plan such a route in which the general scenery and specific landmarks are the fascinating applications of the subject, but we have chosen a different route based on showing mathematics as interesting in its own right. Actually, 'interesting' is rather too bland a word to express the feelings we hope this book will foster; 'excitement' and 'exhilaration' would be more accurate. People find it hard to believe that mathematicians can feel the same sort of emotions and satisfactions for their subject as those with poetic, musical or even sporting inclinations can feel for theirs. Nevertheless it is true, and this has been a factor to consider in planning the route.

Another major thread running throughout the book is the idea of proof.

It is fair to say that the criteria for and methods of proof make mathematics unique among all academic disciplines—even the physical sciences. Sadly, proof is an aspect of mathematical education which is often neglected, and the effect of this neglect can be severe. It means that we sell a subject to prospective students under false pretences, and when they find that university mathematics is very largely about proof and that they don't like it, they have no Trade Descriptions Act to protect them from a fairly miserable three years. There may be a complementary, equally sad effect—that of some potential students never embarking upon a mathematics degree precisely because no one ever demonstrated to them that the search for proof was a pleasure and not a chore of mathematical life, though, for obvious reasons, evidence for this is harder to obtain! To appreciate proof, students must be properly prepared, at sixth-form level and even before—otherwise a subject such as first-year analysis will appear to be all about proving obvious, not very exciting facts by perversely difficult methods. What, then, is an obvious fact, and what do we gain by proving it? Chapter 2 goes some way towards providing an answer.

At the other extreme, there are some unbelievable (yet true) statements about such basics as numbers and graphs, and we plan to highlight these too. Profundity hidden in the obvious and shock and wonder generated by the totally unexpected provide the satisfactions and excitement of mathematics, and it is natural that those who have experienced them should feel some missionary zeal in spreading the word to others, especially as so many have a view of mathematics as 'sums, only harder'. Some who are 'good at mathematics' sometimes try to tell others that the subject is easy. Such people are not mathematicians; they are those who are satisfied with solving problems they find easy anyway! Real satisfaction, whether in marathon running, mountaineering, mathematics or music, comes from accepting the challenge of hard problems, with some exercise of judgement to avoid selecting too many unrealistically difficult ones!

But in mathematics, solving problems is not sufficient. If every problem required the invention of a totally new technique, it would become impossibly difficult rather than just difficult, and satisfaction would rapidly diminish as the feeling that the subject had no intrinsic unity increased. For this reason unity is another important theme. To take account of it, a decision had to be made about exactly which proof techniques to include, and the answer had to be those proofs based on principles which, although very simple, have profound consequences in a variety of different parts of mathematics. We have, in general, avoided one-off tricks except where they are so neat that they cry out for inclusion.

Finally, although the book has, we hope, what computing colleagues would call a reader-friendly style, this is by no means incompatible with mathematical rigour, and we intend to *do* a lot of serious mathematics as well as talk about it! Some of the questions we ask (such as 'What is a real number?' or 'What counts as a proof?') turn out to be profound, and the hard work they entail

is shared between deciding what the significance of the question is and what it would mean to answer it, and doing the detailed working through to achieve the answer. At various points there are gaps indicated thus [m.n] to be filled by the reader, and hints are provided in the appendix. Ends of proofs are signalled by the Halmos tombstone symbol, $\square$, and particularly important results and definitions are enclosed by a box.

The book does not have to be read in the order in which it is written. Depending on the knowledge and interests of the reader, there are convenient places to change the route, and these, too, are signposted. It is important to point out that this is not a systematic textbook on any particular branch of mathematics, but our hope is that its overall effect will be to provide motivation to tackle more formal texts, ability to understand and enjoy them, and a healthy critical attitude which will demand close attention both to rigour and to a sound intuitive base.

‘The purpose of rigour is to legitimate the conquests of intuition’ (Jacques Hadamard, 1865–1962)—and to reveal its not-too-infrequent blunders, we should perhaps add!

We extend our thanks to Macmillan Education’s former mathematics editor, Peter Oates, for his help and patience; to Cathy Crewe, Angela Fullerton and Diane Holloway for transforming illegible scrawl; and to each other for mutual support. As for remaining errors, I blame him and he blames me.

Nottingham and Abingdon, 1987

J.B.
R.H.