GUIDE TO ANALYSIS

Consultant Editor: David A. Towers,
Senior Lecturer in Mathematics,
University of Lancaster

Titles Available

Abstract Algebra
Linear Algebra
Analysis

Further titles are in preparation.

Guide to Analysis

F. Mary Hart

*Department of Pure Mathematics
University of Sheffield*

M
MACMILLAN
EDUCATION

© F. M. Hart 1988

All rights reserved. No reproduction, copy or transmission of this publication may be made without written permission.

No paragraph of this publication may be reproduced, copied or transmitted save with written permission or in accordance with the provisions of the Copyright Act 1956 (as amended), or under the terms of any licence permitting limited copying issued by the Copyright Licensing Agency, 7 Ridgmount Street, London WC1E 7AE.

Any person who does any unauthorised act in relation to this publication may be liable to criminal prosecution and civil claims for damages.

First published 1988

Published by
MACMILLAN EDUCATION LTD
Houndmills, Basingstoke, Hampshire RG21 2XS
and London
Companies and representatives
throughout the world

British Library Cataloguing in Publication Data
Hart, F. Mary

Guide to analysis.—(Macmillan
mathematical guides).

1. Calculus

I. Title

515 QA303

ISBN 978-0-333-43788-9 ISBN 978-1-349-09390-8 (eBook)

DOI 10.1007/978-1-349-09390-8

CONTENTS

Editor's foreword	vii
Preface	viii
Glossary of symbols and notation	x
1 NUMBERS AND NUMBER SYSTEMS	1
1.1 Natural numbers	1
1.2 Integers	2
1.3 Rational numbers	3
1.4 The real number system	7
Appendix: Mathematical induction	20
2 SEQUENCES	25
2.1 Introduction	25
2.2 Sequences	26
2.3 Standard limits	42
2.4 Some general results for sequences	47
2.5 Subsequences	51
Appendix: Triangle inequalities	56
Table of standard limits	58
3 INFINITE SERIES	62
3.1 Introduction	62
3.2 Series and notation	64
3.3 Tests for convergence	69
3.4 Absolute convergence, rearrangement of series, multiplication of series	92
Appendix	100
Tests for convergence—summary	103

4	FUNCTIONS Limits of functions, continuous functions	106
4.1	Limits of functions	106
4.2	Continuity	139
4.3	Some properties of continuous functions on closed intervals	145
5	DIFFERENTIABLE FUNCTIONS Differentiable functions, Rolle's theorem, mean value theorem, Taylor's theorem, l'Hôpital's rule, maxima and minima, Taylor series	160
5.1	Differentiation	160
5.2	Rolle's theorem and the mean value theorem	173
5.3	Taylor's theorem	180
5.4	L'Hôpital's rule	183
5.5	Maxima and minima	187
	Index	201

EDITOR'S FOREWORD

Wide concern has been expressed in tertiary education about the difficulties experienced by students during their first year of an undergraduate course containing a substantial component of mathematics. These difficulties have a number of underlying causes, including the change of emphasis from an algorithmic approach at school to a more rigorous and abstract approach in undergraduate studies, the greater expectation of independent study, and the increased pace at which material is presented. The books in this series are intended to be sensitive to these problems.

Each book is a carefully selected, short, introductory text on a key area of the first-year syllabus; the areas are complementary and largely self-contained. Throughout, the pace of development is gentle, sympathetic and carefully motivated. Clear and detailed explanations are provided, and important concepts and results are stressed.

As mathematics is a practical subject which is best learned by doing it, rather than watching or reading about someone else doing it, a particular effort has been made to include a plentiful supply of worked examples, together with appropriate exercises, ranging in difficulty from the straightforward to the challenging.

When one goes fellwalking, the most breathtaking views require some expenditure of effort in order to gain access to them; nevertheless, the peak is more likely to be reached if a gentle and interesting route is chosen. The mathematical peaks attainable in these books are every bit as exhilarating, the paths are as gentle as we could find, and the interest and expectation are maintained throughout to prevent the spirits from flagging on the journey.

Lancaster, 1987

David A. Towers
Consultant Editor

PREFACE

The transition from sixth form to university or polytechnic can be traumatic. Students have to adapt to different styles of teaching with lectures replacing lessons and at the same time they are often presented with material which is unlike anything they have encountered before. The provision of suitable textbooks can go some way to alleviating the problem. As far as pure mathematics is concerned, analysis, which is central to most undergraduate courses in mathematics, is likely to provide the biggest stumbling-block. The subject itself is intrinsically difficult. The final polished version we have today owes its form to years of concentrated work by many celebrated mathematicians, and students often find it very daunting to be faced with this finished product. They seldom realise just how much effort went into its making, who did the work or why they did it. This account therefore includes some historical notes and anecdotes which help to put the subject into context and, it is hoped, enliven the text for the reader.

This volume is written specifically for students meeting analysis for the first time. It is designed so that it can be used as a textbook in conjunction with a lecture course in analysis or it can be used by students wishing to teach themselves the subject—independent of any lecture course. For this reason numerous worked examples are included with very detailed solutions to enable the reader to check that the concepts and definitions are understood. When working the exercises the reader is given the opportunity to receive a little help (or inspiration!) without being given the answer, as the answers and the hints for solutions are recorded in separate sections. In practice, one exercise successfully completed by the reader (even with the help of a few hints) can teach more than several examples read from the text.

Analysis traditionally contains numerous theorems and proofs. These theorems are the tools of the trade and we cannot just dispense with large numbers of them in order either to cut down the size of the volume or to make the learning process appear easier. Neither can we simply omit their proofs. Before using a result we must know for certain that it is true. The need for proof is something which has always been recognised to a greater or lesser extent. Historical records show that the ancient Greek mathematicians living

centuries before Christ acknowledged this, and, indeed, some of our present-day proofs, like the contradiction argument, stem from their work. However, the use of the link between limits of functions and sequences allows us a few economies and reduces the number of separate proofs necessary.

For many undergraduates the greatest hurdle is the problem posed by the idea of rigour. They are unsure about how much detail needs to be included and how much may reasonably be omitted. The reader of more advanced texts can be assumed to have sufficient judgement to be able to cope when details are not carefully spelled out. A more elementary text, however, needs different treatment and it seems prudent to include all the small details. This may make the definitions and theorems appear a little more cumbersome, but it leaves the reader in no doubt as to what is being assumed. The benefits far outweigh any superficial disadvantage occurring from the longer, more detailed statements. Since repetitions can be valuable for reinforcing knowledge, some statements and proofs which closely resemble earlier ones are nevertheless recorded in detail rather than being dismissed with an assertion of the form 'the proof is similar to that of...'.

Finally, I would like to record my profound thanks for all the secretarial help I have received during the preparation of this volume. My thanks go to Anne Hall and Janet Williams and particularly to Alex Cain who has undertaken the bulk of the work.

Sheffield, 1987

F. M. Hart

GLOSSARY OF SYMBOLS AND NOTATION

$\mathbb{N}$	Set of all natural numbers. This set consists of the numbers 0, 1, 2, 3, ...
$\mathbb{Z}$	Set of all integers
$\mathbb{Z}^+$	Set of all positive integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R} \setminus \{0\}$	Set of all non-zero real numbers
$ x $	Modulus of a real number x . Its value is x if $x \geq 0$ and its value is $-x$ if $x < 0$. Thus $ x = +\sqrt{x^2}$
$>$	Greater than
$<$	Less than
$\nrightarrow$	Not greater than
$\nless$	Not less than
$\geq$	Greater than or equal to
$\leq$	Less than or equal to
$\{x: P(x)\}$	The set of all x such that $P(x)$ holds
$\{a, b, c, \dots\}$	The set consisting of $a, b, c, \dots$
$\rightarrow$	Tends to
$\nrightarrow$	Does not tend to
$\in$	Belongs to
$\notin$	Does not belong to
$(a_n)_{n=1}^{\infty}$	Infinite sequence $a_1, a_2, a_3, \dots$ whose n th term is a_n . (This notation is used when we wish to make clear which values are assumed by the subscript n)
(a_n)	Infinite sequence with terms of the form a_n . (This notation is used when it is clear from the context which values are assumed by n)
$(a_{n_k})_{k=1}^{\infty}$	Subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \dots$ of the sequence (a_n)
$\sum$	Sum of

$\sum_{n=1}^{\infty} a_n$	This is used in two senses. It is used formally to stand for the infinite series $a_1 + a_2 + a_3 + \dots$. It is also used for the sum of the infinite series in the case in which the series is convergent. The context makes clear which meaning is appropriate
$\sum_{k=1}^n a_k$	The sum of the n terms $a_1 + a_2 + \dots + a_n$
γ	Euler's constant
$f: A \rightarrow \mathbb{R}$	Function f from A into $\mathbb{R}$ i.e. f has domain of definition A and range a subset of $\mathbb{R}$
$\log x$	The <i>natural</i> logarithm of x . Throughout this volume all logarithms are to the base e
(a, ∞)	Set of all real numbers x such that $x > a$
$[a, \infty)$	Set of all real numbers x such that $x \geq a$
$(-\infty, b)$	Set of all real numbers x such that $x < b$
$(-\infty, b]$	Set of all real numbers x such that $x \leq b$
(a, b)	The open interval (a, b) . It consists of all the real numbers x such that $a < x < b$
$[a, b]$	The closed interval $[a, b]$. It consists of all the real numbers x such that $a \leq x \leq b$
$x \rightarrow a$	x approaches a (two-sided limit)
$x \rightarrow a^+$	x approaches a through values greater than a (one-sided limit)
$x \rightarrow a^-$	x approaches a through values less than a (one-sided limit)
$g \circ f$	g composition f . This is the function such that $(g \circ f)(x) = g(f(x))$
f^{-1}	Inverse function of f
f'	Derivative of f
IVT	Intermediate value theorem
MVT	Mean value theorem