

Generalised Thermostatistics

Jan Naudts

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Jan Naudts
University of Antwerp
Physics Department
Universiteitsplein 1
2610 Antwerp
Belgium
jan.naudts@ua.ac.be

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Preface

This book is in the first place a monograph, for a large part based on my own research. The same notations are used throughout the text and a strict unity of subject is attempted. On the other hand, care is taken that the book can be used as a textbook for graduate students and as a reference text for researchers.

The point of view adopted here is that statistical physics is statistics applied to physics. The essence of statistical physics is the statistical analysis of models which approximate the physical reality. These models consist of a probability distribution, or its quantum mechanical analogue, used to calculate statistical averages. The probability distribution depends on a small number of parameters which can be estimated from experimental data.

The roots of statistical physics lie in thermodynamics, a nineteenth century science predating statistical physics. One of the goals of statistical physics is to explain thermodynamical concepts in terms of a microscopic theory. But thermodynamics is so generally valid that its relations still hold in a much wider context than that of the traditional theory of statistical mechanics. Therefore, some notions of thermodynamics appear as a skeleton throughout the book.

Recent years, considerable efforts have been made to extend the statistical physics formalism beyond the limits set out by Gibbs [5] in his book, published in 1902. Traditional statistical physics focuses on systems with many degrees of freedoms. The formalism becomes exact in the *thermodynamic limit*, this is, the limit of infinitely many degrees of freedom. One motivation to go beyond the standard formalism is the current interest in relatively small systems. Many new insights originate from *Tsallis' non-extensive thermostatics*, a domain of research that developed during the past twenty years. Let me mention in particular the notions of *deformed exponential and logarithmic functions*, and of *escort probability distributions*, notions that play an important role in Part II of the book.

The title of this book refers to the well-known book by Callen, "Thermodynamics and an Introduction to Thermostatistics" [3], which is often cited

in that part of literature which is concerned with non-extensive thermostatics.

The emphasis in the present work lies on the development of the formalism. For applications of non-extensive statistical physics the reader is referred to the book by Constantino Tsallis [7], to proceedings of conferences [4, 1], and to some review papers [6, 2]. A number of topics, playing a central role in traditional statistical physics, are not treated in the present text. Let me mention the *equivalence of ensembles*, the *thermodynamic limit*, the *central limit theorem*, *large deviation theory*. Time-dependent phenomena are not discussed. The main reason for the latter limitation is that *non-equilibrium statistical physics* is an active area of research with only recent understanding of some, not all, of its fundamentals.

This book does not intend to review the research literature on non-extensive thermostatics. It rather situates this subject in a broader context and aims at consolidating its results. The short notes at the end of each Chapter try to indicate some aspects of the historical developments but fall short of giving proper credit to all researchers active in the field.

I am grateful to all colleagues who helped me to improve the contents of this book. I am especially indebted to Marek Chachor, who introduced me to non-extensive statistical physics.

Antwerpen, September 2010

Jan Naudts

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