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Monomial Ideals

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To our wives Maja and Kumiko, our children Susanne, Ulrike,
Masaki and Ayako, and our grandchildren Paul, Jonathan,
Vincent, Nelson, Sofia and Jesse

Preface

Commutative algebra has developed in step with algebraic geometry and has played an essential role as the foundation of algebraic geometry. On the other hand, homological aspects of modern commutative algebra became a new and important focus of research inspired by the work of Melvin Hochster. In 1975, Richard Stanley [Sta75] proved affirmatively the upper bound conjecture for spheres by using the theory of Cohen–Macaulay rings. This created another new trend of commutative algebra, as it turned out that commutative algebra supplies basic methods in the algebraic study of combinatorics on convex polytopes and simplicial complexes. Stanley was the first to use concepts and techniques from commutative algebra in a systematic way to study simplicial complexes by considering the Hilbert function of Stanley–Reisner rings, whose defining ideals are generated by squarefree monomials. Since then, the study of squarefree monomial ideals from both the algebraic and combinatorial points of view has become a very active area of research in commutative algebra.

In the late 1980s the theory of Gröbner bases came into fashion in many branches of mathematics. Gröbner bases, together with initial ideals, provided new methods. They have been used not only for computational purposes but also to deduce theoretical results in commutative algebra and combinatorics. For example, based on the fundamental work by Gel'fand, Kapranov, Zelevinsky and Sturmfels, far beyond the classical techniques in combinatorics, the study of regular triangulations of a convex polytope by using suitable initial ideals turned out to be a very successful approach, and, after the pioneering work of Sturmfels [Stu90], the algebraic properties of determinantal ideals have been explored by considering their initial ideal, which for a suitable monomial order is a squarefree monomial ideal and hence is accessible to powerful techniques.

At about the same time Galligo, Bayer and Stillman observed that generic initial ideals have particularly nice combinatorial structures and provide a basic tool for the combinatorial and computational study of the minimal free resolution of a graded ideal of the polynomial ring. Algebraic shifting, which was introduced by Kalai and which contributed to the modern development of enumerative combinatorics on simplicial complexes, can be discussed in the frame of generic initial ideals.

The present monograph invites the reader to become acquainted with current trends in combinatorial commutative algebra, with the main emphasis on basic research into monomials and monomial ideals. Apart from a few exceptions, where we refer to the books [BH98], [Kun08] and [Mat80], only basic knowledge of commutative algebra is required to understand most of the monograph. Part I is a self-contained introduction to the modern theory of Gröbner bases and initial ideals. Its highlight is a quick introduction to the theory of Gröbner bases (Chapter 2), and it also offers a detailed description of, and information about, generic initial ideals (Chapter 4). Part II covers Hilbert functions and resolutions and some of the combinatorics related to monomial ideals, including the Kruskal–Katona theorem and algebraic aspects of Alexander duality. In Part III we discuss combinatorial applications of monomial ideals. The main topics include edge ideals of finite graphs, powers of ideals, algebraic shifting theory and an introduction to polymatroids.

We now discuss the contents of the monograph in detail together with a brief history of commutative algebra and combinatorics on monomials and monomial ideals.

Chapter 1 summarizes fundamental material on monomial ideals. In particular, we consider the integral closure of monomial ideals, squarefree normally torsionfree ideals, squarefree monomial ideals and simplicial complexes, Alexander duality and polarization of monomial ideals.

In Chapter 2 a short introduction to the main features of Gröbner basis theory is given, including the Buchberger criterion and algorithm. These basic facts are discussed in a comprehensive but compact form.

Chapter 3 presents one of the most fundamental results on initial ideals, which says that the graded Betti numbers of the initial ideal $\mathrm{in}_<(I)$ are greater than or equal to the corresponding graded Betti numbers of I. This fact is used again and again in this book, especially in shifting theory.

Chapter 4 concerns generic initial ideals. This theory plays an essential role in the combinatorial applications considered in Part III. Therefore, for the sake of completeness, we present in Chapter 4 the main theorems on generic initial ideals together with their complete proofs. Generic initial ideals are Borel-fixed. They belong to the more general class of Borel type ideals for which various characterizations are given. Generic annihilator numbers and extremal Betti numbers are introduced, and it is shown that extremal Betti numbers are invariant under taking generic initial ideals.

Chapter 5 is devoted to establishing the theory of Gröbner bases in the exterior algebra, and uses exterior techniques to give a proof of the Alexander duality theorem which establishes isomorphisms between simplicial homology and cohomology of a simplicial complex and its Alexander dual.

Chapter 6 offers basic material on combinatorics of monomial ideals. First we recall the concepts of Hilbert functions and Hilbert polynomials, and their relationship to the f-vector of a simplicial complex is explained. We study in detail the combinatorial characterization of Hilbert functions of graded ideals due to Macaulay together with its squarefree analogue, the Kruskal–Katona theorem, which describes the possible face numbers of simplicial complexes. Lexsegment ideals as well as squarefree lexsegment ideals play the key role in the discussion.

Chapter 7 discusses minimal free resolutions of monomial ideals. We derive formulas for the graded Betti numbers of stable and squarefree stable ideals, and use these formulas to deduce the Bigatti–Hulett theorem which says that lexsegment ideals have the largest graded Betti numbers among all graded ideals with the same Hilbert function. We also present the squarefree analogue of the Bigatti–Hulett theorem, and give the comparison of Betti numbers over the symmetric and exterior algebra.

Chapter 8 begins with Hochster's formula to compute the graded Betti numbers of Stanley–Reisner ideals and Reisner's Cohen–Macaulay criterion for simplicial complexes. Then the Eagon–Reiner theorem and variations of it are discussed. In particular, ideals with linear quotients, componentwise linear ideals, sequentially Cohen–Macaulay ideals and shellable simplicial complexes are studied.

Chapter 9 deals with the algebraic aspects of Dirac's theorem on chordal graphs and the classification problem for Cohen–Macaulay graphs. First the classification of bipartite Cohen–Macaulay graphs is given. Then unmixed graphs are characterized and we present the result which says that a bipartite graph is sequentially Cohen–Macaulay if and only if it is shellable. It follows the classification of Cohen–Macaulay chordal graphs. Finally the relationship between the Hilbert–Burch theorem and Dirac's theorem on chordal graphs is explained.

Chapter 10 is devoted to the study of powers of monomial ideals. We begin with a brief introduction to toric ideals and Rees algebras, and present a Gröbner basis criterion which guarantees that all powers of an ideal have a linear resolution. As an application it is shown that all powers of monomial ideals with 2-linear resolution have a linear resolution. Then the depth of powers of monomial ideals, and Mengerian and unimodular simplicial complexes are considered.

Chapter 11 offers a self-contained and systematic presentation of modern shifting theory from the viewpoint of generic initial ideals as well as of graded Betti numbers. Combinatorial, exterior and symmetric shifting are introduced and the comparison of the graded Betti numbers for the distinct shifting operators is studied. It is shown that the extremal graded Betti numbers of a simplicial complex and its symmetric and exterior shifted complex are the same. Finally, super-extremal Betti numbers are considered to give an algebraic proof of the Björner–Kalai theorem.

In Chapter 12 we consider discrete polymatroids and polymatroidal ideals. After giving a short introduction to the combinatorics and geometry of discrete polymatroids, the algebraic properties of its base ring are studied. We close

Chapter 12 by presenting polymatroidal and weakly polymatroidal ideals, which provide large classes of ideals with linear quotients.

It becomes apparent from the above detailed description of the topics discussed in this monograph that the authors have chosen those combinatorial topics which are strongly related to monomial ideals. Binomial ideals, toric rings and convex polytopes are not the main topic of this book. We refer the reader to Sturmfels [Stu96], Miller–Sturmfels [MS04] and Bruns–Gubeladze [BG09]. We also do not discuss the pioneering work by Richard Stanley on the upper bound conjecture for spheres. For this topic we refer the reader to Bruns–Herzog [BH98], Hibi [Hib92] and Stanley [Sta95].

We have tried as much as possible to make our presentation self-contained, and we believe that combinatorialists who are familiar with only basic materials on commutative algebra will understand most of this book without having to read other textbooks or research papers. However, for the convenience of the reader who is not so familiar with commutative algebra and convex geometry we have added an appendix in which we explain some fundamental algebraic and geometric concepts which are used in this book. In addition, researchers working on applied mathematics who want to learn Gröbner basis theory quickly as a basic tool for their work need only consult Chapter 2. Since shifting theory is rather technical, the reader may skip Chapters 4–7 and 11 (which are required for the understanding of shifting theory) on a first reading.

We conclude each chapter with a list of problems. They are intended to complement and provide better understanding of the topics treated in each chapter.

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