

Graduate Texts in Mathematics 260

Editorial Board

S. Axler

K.A. Ribet

For other titles published in this series, go to
www.springer.com/series/136

Jürgen Herzog • Takayuki Hibi

Monomial Ideals

 Springer

Jürgen Herzog
Universität Duisburg-Essen
Fachbereich Mathematik
Campus Essen
Universitätsstraße 2
D-45141 Essen
Germany
juergen.herzog@uni-due.de

Takayuki Hibi
Department of Pure
and Applied Mathematics
Graduate School of Information Science
and Technology
Osaka University
Toyonaka, Osaka 560-0043
Japan
hibi@math.sci.osaka-u.ac.jp

Editorial Board

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA
axler@sfsu.edu

K.A. Ribet
Mathematics Department
University of California, Berkeley
Berkeley, CA 94720-3840
USA
ribet@math.berkeley.edu

ISSN 0072-5285
ISBN 978-0-85729-105-9
DOI 10.1007/978-0-85729-106-6
Springer London Dordrecht Heidelberg New York

e-ISBN 978-0-85729-106-6

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2010937479

Mathematics Subject Classification (2010): 13D02, 13D40, 13F55, 13H10, 13P10

© Springer-Verlag London Limited 2011

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms of licenses issued by the Copyright Licensing Agency. Enquiries concerning reproduction outside those terms should be sent to the publishers. The use of registered names, trademarks, etc., in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant laws and regulations and therefore free for general use.

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

Cover design: VTEX, Vilnius

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To our wives Maja and Kumiko, our children Susanne, Ulrike,
Masaki and Ayako, and our grandchildren Paul, Jonathan,
Vincent, Nelson, Sofia and Jesse

Preface

Commutative algebra has developed in step with algebraic geometry and has played an essential role as the foundation of algebraic geometry. On the other hand, homological aspects of modern commutative algebra became a new and important focus of research inspired by the work of Melvin Hochster. In 1975, Richard Stanley [Sta75] proved affirmatively the upper bound conjecture for spheres by using the theory of Cohen–Macaulay rings. This created another new trend of commutative algebra, as it turned out that commutative algebra supplies basic methods in the algebraic study of combinatorics on convex polytopes and simplicial complexes. Stanley was the first to use concepts and techniques from commutative algebra in a systematic way to study simplicial complexes by considering the Hilbert function of Stanley–Reisner rings, whose defining ideals are generated by squarefree monomials. Since then, the study of squarefree monomial ideals from both the algebraic and combinatorial points of view has become a very active area of research in commutative algebra.

In the late 1980s the theory of Gröbner bases came into fashion in many branches of mathematics. Gröbner bases, together with initial ideals, provided new methods. They have been used not only for computational purposes but also to deduce theoretical results in commutative algebra and combinatorics. For example, based on the fundamental work by Gel'fand, Kapranov, Zelevinsky and Sturmfels, far beyond the classical techniques in combinatorics, the study of regular triangulations of a convex polytope by using suitable initial ideals turned out to be a very successful approach, and, after the pioneering work of Sturmfels [Stu90], the algebraic properties of determinantal ideals have been explored by considering their initial ideal, which for a suitable monomial order is a squarefree monomial ideal and hence is accessible to powerful techniques.

At about the same time Galligo, Bayer and Stillman observed that generic initial ideals have particularly nice combinatorial structures and provide a basic tool for the combinatorial and computational study of the minimal free resolution of a graded ideal of the polynomial ring. Algebraic shifting, which was introduced by Kalai and which contributed to the modern development

of enumerative combinatorics on simplicial complexes, can be discussed in the frame of generic initial ideals.

The present monograph invites the reader to become acquainted with current trends in combinatorial commutative algebra, with the main emphasis on basic research into monomials and monomial ideals. Apart from a few exceptions, where we refer to the books [BH98], [Kun08] and [Mat80], only basic knowledge of commutative algebra is required to understand most of the monograph. Part I is a self-contained introduction to the modern theory of Gröbner bases and initial ideals. Its highlight is a quick introduction to the theory of Gröbner bases (Chapter 2), and it also offers a detailed description of, and information about, generic initial ideals (Chapter 4). Part II covers Hilbert functions and resolutions and some of the combinatorics related to monomial ideals, including the Kruskal–Katona theorem and algebraic aspects of Alexander duality. In Part III we discuss combinatorial applications of monomial ideals. The main topics include edge ideals of finite graphs, powers of ideals, algebraic shifting theory and an introduction to polymatroids.

We now discuss the contents of the monograph in detail together with a brief history of commutative algebra and combinatorics on monomials and monomial ideals.

Chapter 1 summarizes fundamental material on monomial ideals. In particular, we consider the integral closure of monomial ideals, squarefree normally torsionfree ideals, squarefree monomial ideals and simplicial complexes, Alexander duality and polarization of monomial ideals.

In Chapter 2 a short introduction to the main features of Gröbner basis theory is given, including the Buchberger criterion and algorithm. These basic facts are discussed in a comprehensive but compact form.

Chapter 3 presents one of the most fundamental results on initial ideals, which says that the graded Betti numbers of the initial ideal $\text{in}_{<}(I)$ are greater than or equal to the corresponding graded Betti numbers of I . This fact is used again and again in this book, especially in shifting theory.

Chapter 4 concerns generic initial ideals. This theory plays an essential role in the combinatorial applications considered in Part III. Therefore, for the sake of completeness, we present in Chapter 4 the main theorems on generic initial ideals together with their complete proofs. Generic initial ideals are Borel-fixed. They belong to the more general class of Borel type ideals for which various characterizations are given. Generic annihilator numbers and extremal Betti numbers are introduced, and it is shown that extremal Betti numbers are invariant under taking generic initial ideals.

Chapter 5 is devoted to establishing the theory of Gröbner bases in the exterior algebra, and uses exterior techniques to give a proof of the Alexander duality theorem which establishes isomorphisms between simplicial homology and cohomology of a simplicial complex and its Alexander dual.

Chapter 6 offers basic material on combinatorics of monomial ideals. First we recall the concepts of Hilbert functions and Hilbert polynomials, and their relationship to the f -vector of a simplicial complex is explained. We study in

detail the combinatorial characterization of Hilbert functions of graded ideals due to Macaulay together with its squarefree analogue, the Kruskal–Katona theorem, which describes the possible face numbers of simplicial complexes. Lexsegment ideals as well as squarefree lexsegment ideals play the key role in the discussion.

Chapter 7 discusses minimal free resolutions of monomial ideals. We derive formulas for the graded Betti numbers of stable and squarefree stable ideals, and use these formulas to deduce the Bigatti–Hulett theorem which says that lexsegment ideals have the largest graded Betti numbers among all graded ideals with the same Hilbert function. We also present the squarefree analogue of the Bigatti–Hulett theorem, and give the comparison of Betti numbers over the symmetric and exterior algebra.

Chapter 8 begins with Hochster’s formula to compute the graded Betti numbers of Stanley–Reisner ideals and Reisner’s Cohen–Macaulay criterion for simplicial complexes. Then the Eagon–Reiner theorem and variations of it are discussed. In particular, ideals with linear quotients, componentwise linear ideals, sequentially Cohen–Macaulay ideals and shellable simplicial complexes are studied.

Chapter 9 deals with the algebraic aspects of Dirac’s theorem on chordal graphs and the classification problem for Cohen–Macaulay graphs. First the classification of bipartite Cohen–Macaulay graphs is given. Then unmixed graphs are characterized and we present the result which says that a bipartite graph is sequentially Cohen–Macaulay if and only if it is shellable. It follows the classification of Cohen–Macaulay chordal graphs. Finally the relationship between the Hilbert–Burch theorem and Dirac’s theorem on chordal graphs is explained.

Chapter 10 is devoted to the study of powers of monomial ideals. We begin with a brief introduction to toric ideals and Rees algebras, and present a Gröbner basis criterion which guarantees that all powers of an ideal have a linear resolution. As an application it is shown that all powers of monomial ideals with 2-linear resolution have a linear resolution. Then the depth of powers of monomial ideals, and Mengerian and unimodular simplicial complexes are considered.

Chapter 11 offers a self-contained and systematic presentation of modern shifting theory from the viewpoint of generic initial ideals as well as of graded Betti numbers. Combinatorial, exterior and symmetric shifting are introduced and the comparison of the graded Betti numbers for the distinct shifting operators is studied. It is shown that the extremal graded Betti numbers of a simplicial complex and its symmetric and exterior shifted complex are the same. Finally, super-extremal Betti numbers are considered to give an algebraic proof of the Björner–Kalai theorem.

In Chapter 12 we consider discrete polymatroids and polymatroidal ideals. After giving a short introduction to the combinatorics and geometry of discrete polymatroids, the algebraic properties of its base ring are studied. We close

Chapter 12 by presenting polymatroidal and weakly polymatroidal ideals, which provide large classes of ideals with linear quotients.

It becomes apparent from the above detailed description of the topics discussed in this monograph that the authors have chosen those combinatorial topics which are strongly related to monomial ideals. Binomial ideals, toric rings and convex polytopes are not the main topic of this book. We refer the reader to Sturmfels [Stu96], Miller–Sturmfels [MS04] and Bruns–Gubeladze [BG09]. We also do not discuss the pioneering work by Richard Stanley on the upper bound conjecture for spheres. For this topic we refer the reader to Bruns–Herzog [BH98], Hibi [Hib92] and Stanley [Sta95].

We have tried as much as possible to make our presentation self-contained, and we believe that combinatorialists who are familiar with only basic materials on commutative algebra will understand most of this book without having to read other textbooks or research papers. However, for the convenience of the reader who is not so familiar with commutative algebra and convex geometry we have added an appendix in which we explain some fundamental algebraic and geometric concepts which are used in this book. In addition, researchers working on applied mathematics who want to learn Gröbner basis theory quickly as a basic tool for their work need only consult Chapter 2. Since shifting theory is rather technical, the reader may skip Chapters 4–7 and 11 (which are required for the understanding of shifting theory) on a first reading.

We conclude each chapter with a list of problems. They are intended to complement and provide better understanding of the topics treated in each chapter.

We are grateful to Viviana Ene and Rahim Zaare-Nahandi for their comments and for suggesting corrections in some earlier drafts of this monograph.

Essen, Osaka
February 2010

Jürgen Herzog
Takayuki Hibi

Contents

Part I Gröbner bases

1	Monomial Ideals	3
1.1	Basic properties of monomial ideals	3
1.1.1	The K -basis of a monomial ideal	3
1.1.2	Monomial generators	5
1.1.3	The \mathbb{Z}^n -grading	5
1.2	Algebraic operations on monomial ideals	6
1.2.1	Standard algebraic operations	6
1.2.2	Saturation and radical	7
1.3	Primary decomposition and associated prime ideals	8
1.3.1	Irreducible monomial ideals	8
1.3.2	Primary decompositions	10
1.4	Integral closure of ideals	12
1.4.1	Integral closure of monomial ideals	12
1.4.2	Normally torsionfree squarefree monomial ideals	14
1.5	Squarefree monomial ideals and simplicial complexes	15
1.5.1	Simplicial complexes	15
1.5.2	Stanley–Reisner ideals and facet ideals	16
1.5.3	The Alexander dual	17
1.6	Polarization	18
	Problems	20
	Notes	21
2	A short introduction to Gröbner bases	23
2.1	Dickson’s lemma and Hilbert’s basis theorem	23
2.1.1	Dickson’s lemma	23
2.1.2	Monomial orders	24
2.1.3	Gröbner bases	25
2.1.4	Hilbert’s basis theorem	27
2.2	The division algorithm	28

2.2.1	The division algorithm	28
2.2.2	Reduced Gröbner bases	32
2.3	Buchberger's criterion	33
2.3.1	S-polynomials	33
2.3.2	Buchberger's criterion	33
2.3.3	Buchberger's algorithm	37
	Problems	39
	Notes	40
3	Monomial orders and weights	41
3.1	Initial terms with respect to weights	41
3.1.1	Gradings defined by weights	41
3.1.2	Initial ideals given by weights	42
3.2	The initial ideal as the special fibre of a flat family	43
3.2.1	Homogenization	43
3.2.2	A one parameter flat family	44
3.3	Comparison of I and $\text{in}(I)$	45
	Problems	49
	Notes	50
4	Generic initial ideals	51
4.1	Existence	51
4.2	Stability properties of generic initial ideals	55
4.2.1	The theorem of Galligo and Bayer–Stillman	55
4.2.2	Borel-fixed monomial ideals	57
4.3	Extremal Betti numbers	61
4.3.1	Almost regular sequences and generic annihilator numbers	61
4.3.2	Annihilator numbers and Betti numbers	68
	Problems	72
	Notes	73
5	The exterior algebra	75
5.1	Graded modules over the exterior algebra	75
5.1.1	Basic concepts	75
5.1.2	The exterior face ring of a simplicial complex	76
5.1.3	Duality	77
5.1.4	Simplicial homology	80
5.2	Gröbner bases	83
5.2.1	Monomial orders and initial ideals	84
5.2.2	Buchberger's criterion	86
5.2.3	Generic initial ideals and generic annihilator numbers in the exterior algebra	91
	Problems	92
	Notes	93

Part II Hilbert functions and resolutions

6	Hilbert functions and the theorems of Macaulay and Kruskal–Katona	97
6.1	Hilbert functions, Hilbert series and Hilbert polynomials	97
6.1.1	The Hilbert function of a graded R -module	97
6.1.2	Hilbert functions and initial ideals	99
6.1.3	Hilbert functions and resolutions	100
6.2	The h -vector of a simplicial complex	101
6.3	Lexsegment ideals and Macaulay’s theorem	102
6.4	Squarefree lexsegment ideals and the Kruskal–Katona Theorem	109
	Problems	111
	Notes	113
7	Resolutions of monomial ideals and the Eliahou–Kervaire formula	115
7.1	The Taylor complex	115
7.2	Betti numbers of stable monomial ideals	117
7.2.1	Modules with maximal Betti numbers	117
7.2.2	Stable monomial ideals	119
7.3	The Bigatti–Hulett theorem	121
7.4	Betti numbers of squarefree stable ideals	122
7.5	Comparison of Betti numbers over the symmetric and exterior algebra	125
	Problems	127
	Notes	128
8	Alexander duality and resolutions	129
8.1	The Eagon–Reiner theorem	129
8.1.1	Hochster’s formula	129
8.1.2	Reisner’s criterion and the Eagon–Reiner theorem	132
8.2	Componentwise linear ideals	134
8.2.1	Ideals with linear quotients	134
8.2.2	Monomial ideals with linear quotients and shellable simplicial complexes	135
8.2.3	Componentwise linear ideals	139
8.2.4	Ideals with linear quotients and componentwise linear ideals	141
8.2.5	Squarefree componentwise linear ideals	143
8.2.6	Sequentially Cohen–Macaulay complexes	144
8.2.7	Ideals with stable Betti numbers	145
	Problems	147
	Notes	149

Part III Combinatorics

9	Alexander duality and finite graphs	153
9.1	Edge ideals of finite graphs	153
9.1.1	Basic definitions	153
9.1.2	Finite partially ordered sets	156
9.1.3	Cohen–Macaulay bipartite graphs	160
9.1.4	Unmixed bipartite graphs	163
9.1.5	Sequentially Cohen–Macaulay bipartite graphs	165
9.2	Dirac’s theorem on chordal graphs	167
9.2.1	Edge ideals with linear resolution	167
9.2.2	The Hilbert–Burch theorem for monomial ideals	169
9.2.3	Chordal graphs and quasi-forests	172
9.2.4	Dirac’s theorem on chordal graphs	175
9.3	Edge ideals of chordal graphs	176
9.3.1	Cohen–Macaulay chordal graphs	176
9.3.2	Chordal graphs are shellable	180
	Problems	181
	Notes	182
10	Powers of monomial ideals	183
10.1	Toric ideals and Rees algebras	183
10.1.1	Toric ideals	183
10.1.2	Rees algebras and the x -condition	186
10.2	Powers of monomial ideals with linear resolution	189
10.2.1	Monomial ideals with 2-linear resolution	190
10.2.2	Powers of monomial ideals with 2-linear resolution	191
10.2.3	Powers of vertex cover ideals of Cohen–Macaulay bipartite graphs	195
10.2.4	Powers of vertex cover ideals of Cohen–Macaulay chordal graphs	196
10.3	Depth and normality of powers of monomial ideals	197
10.3.1	The limit depth of a graded ideal	197
10.3.2	The depth of powers of certain classes of monomial ideals	199
10.3.3	Normally torsionfree squarefree monomial ideals and Mengerian simplicial complexes	203
10.3.4	Classes of Mengerian simplicial complexes	205
	Problems	207
	Notes	209

11	Shifting theory	211
	11.1 Combinatorial shifting	211
	11.1.1 Shifting operations	211
	11.1.2 Combinatorial shifting	211
	11.2 Exterior and symmetric shifting	212
	11.2.1 Exterior algebraic shifting	213
	11.2.2 Symmetric algebraic shifting	213
	11.3 Comparison of Betti numbers	217
	11.3.1 Graded Betti numbers of I_Δ and I_{Δ^*}	218
	11.3.2 Graded Betti numbers of I_{Δ^e} and I_{Δ^c}	218
	11.3.3 Graded Betti numbers of I_Δ and I_{Δ^e}	221
	11.4 Extremal Betti numbers and algebraic shifting	225
	11.5 Superextremal Betti numbers	231
	Problems	235
	Notes	235
12	Discrete Polymatroids	237
	12.1 Classical polyhedral theory on polymatroids	237
	12.2 Matroids and discrete polymatroids	241
	12.3 Integral polymatroids and discrete polymatroids	245
	12.4 The symmetric exchange theorem	248
	12.5 The base ring of a discrete polymatroid	250
	12.6 Polymatroidal ideals	255
	12.7 Weakly polymatroidal ideals	258
	Problems	259
	Notes	261
A	Some homological algebra	263
	A.1 The language of categories and functors	263
	A.2 Graded free resolutions	265
	A.3 The Koszul complex	268
	A.4 Depth	272
	A.5 Cohen–Macaulay modules	273
	A.6 Gorenstein rings	274
	A.7 Local cohomology	277
	A.8 The Cartan complex	279
B	Geometry	285
	B.1 Convex polytopes	285
	B.2 Linear programming	287
	B.3 Vertices of polymatroids	288
	B.4 Intersection Theorem	291
	B.5 Polymatroidal Sums	291
	B.6 Toric rings	292

References	295
Index	301