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Vincenzo Capasso • David Bakstein

An Introduction to Continuous-Time Stochastic Processes

Theory, Models, and Applications to Finance, Biology, and Medicine

Second Edition

Birkhäuser

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Preface to the Second Edition

In this second edition, we have included additional material for use in modern applications of stochastic calculus in finance and biology; in particular, the section on infinitely divisible distributions and stable laws in Chap. 1, Lévy processes in Chap. 2, the Itô–Lévy calculus in Chap. 3, and Chap. 4. Finally, a new appendix has been added that includes basic facts about semigroups of linear operators.

We have also made an effort to improve the presentation of parts already included in the first edition, and we have corrected the misprints and errors we have been made aware of by colleagues and students during class use of the book in the intervening years. We are very grateful to all those who helped us in detecting them and suggested possible improvements. We are very grateful to Giacomo Aletti, Enea Bongiorno, Daniela Morale, Stefania Ugolini, and Elena Villa for checking the final proofs and suggesting valuable changes.

Enea Bongiorno deserves special mention for his accurate editing of the book as you now see it.

Tom Grasso from Birkhäuser deserves acknowledgement for encouraging the preparation of a second, updated edition.

Milan, Italy

Vincenzo Capasso
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Preface to the First Edition

This book is a systematic, rigorous, and self-contained introduction to the theory of continuous-time stochastic processes. But it is neither a tract nor a recipe book as such; rather, it is an account of fundamental concepts as they appear in relevant modern applications and the literature. We make no pretense of its being complete. Indeed, we have omitted many results that we feel are not directly related to the main theme or that are available in easily accessible sources. Readers interested in the historical development of the subject cannot ignore the volume edited by Wax (1954).

Proofs are often omitted as technicalities might distract the reader from a conceptual approach. They are produced whenever they might serve as a guide to the introduction of new concepts and methods to the applications; otherwise, explicit references to standard literature are provided. A mathematically oriented student may find it interesting to consider proofs as exercises.

The scope of the book is profoundly educational, related to modeling real-world problems with stochastic methods. The reader becomes critically aware of the concepts involved in current applied literature and is, moreover, provided with a firm foundation of mathematical techniques. Intuition is always supported by mathematical rigor.

Our book addresses three main groups of readers: first, mathematicians working in a different field; second, other scientists and professionals from a business or academic background; third, graduate or advanced undergraduate students of a quantitative subject related to stochastic theory or applications.

As stochastic processes (compared to other branches of mathematics) are relatively new, yet increasingly popular in terms of current research output and applications, many pure as well as applied deterministic mathematicians have become interested in learning about the fundamentals of stochastic theory and modern applications. This book is written in a language that both groups will understand and in its content and structure will allow them to learn the essentials profoundly and in a time-efficient manner. Other scientist-practitioners and academics from fields like finance, biology, and medicine might be very familiar with a less mathematical approach to their specific fields and thus be interested in learning the mathematical techniques of modeling their applications.

Furthermore, this book would be suitable as a textbook accompanying a graduate or advanced undergraduate course or as secondary reading for students of mathematical or computational sciences. The book has evolved from course material that has already been tested for many years in various courses in engineering, biomathematics, industrial mathematics, and mathematical finance.

Last, but certainly not least, this book should also appeal to anyone who would like to learn about the mathematics of stochastic processes. The reader will see that previous exposure to probability, though helpful, is not essential and that the fundamentals of measure and integration are provided in a self-contained way. Only familiarity with calculus and some analysis is required.

The book is divided into three main parts. In Part I, comprising Chaps. 1–4, we introduce the foundations of the mathematical theory of stochastic processes and stochastic calculus, thereby providing the tools and methods needed in Part II (Chaps. 5 and 6), which is dedicated to major scientific areas of application. The third part consists of appendices, each of which gives a basic introduction to a particular field of fundamental mathematics (e.g., measure, integration, metric spaces) and explains certain problems in greater depth (e.g., stability of ODEs) than would be appropriate in the main part of the text.

In Chap. 1 the fundamentals of probability are provided following a standard approach based on Lebesgue measure theory due to Kolmogorov. Here the guiding textbook on the subject is the excellent monograph by Métivier (1968). Basic concepts from Lebesgue measure theory are also provided in Appendix A.

Chapter 2 gives an introduction to the mathematical theory of stochastic processes in continuous time, including basic definitions and theorems on processes with independent increments, martingales, and Markov processes. The two fundamental classes of processes, Poisson and Wiener, are introduced as well as the larger, more general, class of Lévy processes. Further, a significant introduction to marked point processes is also given as a support for the analysis of relevant applications.

Chapter 3 is based on Itô theory. We define the Itô integral, some fundamental results of Itô calculus, and stochastic differentials including Itô's formula, as well as related results like the martingale representation theorem.

Chapter 4 is devoted to the analysis of stochastic differential equations driven by Wiener processes and Itô diffusions and demonstrates the connections with partial differential equations of second order, via Dynkin and Feynman–Kac formulas.

Chapter 5 is dedicated to financial applications. It covers the core economic concept of arbitrage-free markets and shows the connection with martingales and Girsanov's theorem. It explains the standard Black–Scholes theory and relates it to Kolmogorov's partial differential equations and the Feynman–Kac formula. Furthermore, extensions and variations of the standard theory are discussed as are interest rate models and insurance mathematics.

Chapter 6 presents fundamental models of population dynamics such as birth and death processes. Furthermore, it deals with an area of important modern research—the fundamentals of self-organizing systems, in particular focusing on the social behavior of multiagent systems, with some applications to economics ("price herding"). It also includes a particular application to the neurosciences, illustrating the importance of stochastic differential equations driven by both Poisson and Wiener processes.

Problems and additions are proposed at the end of the volume, listed by chapter. In addition to exercises presented in a classical way, problems are proposed as a stimulus for discussing further concepts that might be of interest to the reader. Various sources have been used, including a selection of problems submitted to our students over the years. This is why we can provide only selected references.

The core of this monograph, on Itô calculus, was developed during a series of courses that one of the authors, VC, has been offering at various levels in many universities. That author wishes to acknowledge that the first drafts of the relevant chapters were the outcome of a joint effort by many participating students: Maria Chiarolla, Luigi De Cesare, Marcello De Giosa, Lucia Maddalena, and Rosamaria Mininni, among others. Professor Antonio Fasano is due our thanks for his continuous support, including producing such material as lecture notes within a series that he coordinated.

It was the success of these lecture notes, and the particular enthusiasm of coauthor DB, who produced the first English version (indeed, an unexpected Christmas gift), that has led to an extension of the material up to the present status, including, in particular, a set of relevant and updated applications that reflect the interests of the two authors.

VC would also like to thank his first advisor and teacher, Professor Grace Yang, who gave him the first rigorous presentation of stochastic processes and mathematical statistics at the University of Maryland at College Park, always referring to real-world applications. DB would like to thank the Meregalli and Silvestri families for their kind logistical help while he was in Milan. He would also like to acknowledge research funding from the EPSRC, ESF, Socrates–Erasmus, and Charterhouse and thank all the people he worked with at OCIAM, University of Oxford, over the years, as this is where he was based when embarking on this project.

The draft of the final volume was carefully read by Giacomo Aletti, Daniela Morale, Alessandra Micheletti, Matteo Ortisi, and Enea Bongiorno (who also took care of the problems and additions) whom we gratefully acknowledge. Still, we are sure that some odd typos and other, hopefully noncrucial, mistakes remain, for which the authors take full responsibility.

We also wish to thank Professor Nicola Bellomo, editor of the "Modeling and Simulation in Science, Engineering and Technology" series, and Tom Grasso from Birkhäuser for supporting the project. Last but not least, we

cannot neglect to thank Rossana (VC) and Casilda (DB) for their patience and great tolerance while coping with their “solitude” during the preparation of this monograph.

Milan, Italy

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David Bakstein

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