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Introduction to Vertex Operator Algebras and Their Representations

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To Lael and Mei

To Dongyuan, Joyce and Jimmy

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Preface

Vertex operator algebra theory is a new area of mathematics. It has been an exciting and ever-growing subject from the beginning, starting even before R. Borcherds introduced the precise mathematical notion of "vertex algebra" in the 1980s [B1]. Having developed in conjunction with string theory in theoretical physics and with the theory of "monstrous moonshine" and infinite-dimensional Lie algebra theory in mathematics, vertex (operator) algebra theory is qualitatively different from traditional algebraic theories, reflecting the "nonclassical" nature of string theory and of monstrous moonshine. The theory has revealed new perspectives that were unavailable without it, and continues to do so.

"Monstrous moonshine" began as an astonishing set of conjectures relating the Monster finite simple group to the theory of modular functions in number theory. As is now known, vertex operator algebra theory is a foundational pillar of monstrous moonshine. With the theory available, one can formulate and try to solve new problems that have far-reaching implications in a wide range of fields that had not previously been thought of as being related.

This book systematically introduces the theory of vertex (operator) algebras from the beginning, using "formal calculus," and takes the reader through the fundamental theory to the detailed construction of examples. The axiomatic foundations of vertex operator algebras and modules are studied in detail, general construction theorems for vertex operator algebras and modules are presented, and the most basic families of vertex operator algebras are constructed and their irreducible modules are constructed and are also classified. The construction theorems for algebras and modules are based on a study of *representations* of a vertex operator algebra, as opposed to *modules* for the algebra, as developed in [Li3]. A significant feature of the theory is that in general, the construction of modules for (or representations of) a vertex operator algebra is in some sense more subtle than the construction of the algebra itself. With the body of theory presented in this book as background, the reader will be well prepared to embark on any of a vast range of directions in the theory and its applications.

In the introduction, we shall sketch the theory and the contents of this book and provide motivation. We have written this book to be self-contained. The only prerequisites are some familiarity with Lie algebras and a desire to learn vertex operator algebra theory. If the reader can follow the mathematical material in the introduction, then he or she will have no trouble following the whole book.

This book is suitable for a first graduate course on vertex operator algebra theory, and in fact, that is how we have been using drafts of this book while writing and expanding it over the years.

As we have mentioned, our treatment is axiomatic, leading to the construction of examples. By contrast, the original book [FLM6] developed examples and the theory of *vertex operators* before presenting the concept of *vertex operator algebra* as a "conceptual summary" of the fundamental properties that had been built up through an ever-deepening study of certain basic structures (structures that were needed in the construction of the "moonshine module" for the Monster finite simple group in [FLM6]). That book is not a prerequisite for this book, but the reader may wish to consult [FLM6] to get a feeling for how vertex operator algebra theory originally developed through examples and through the solution of problems.

We continue to use the historical terms "vertex algebra," which was the original term of Borcherds, and "vertex operator algebra," which was used in [FLM6] for a certain variant notion. (Both notions are discussed at great length in this book.) A vertex operator algebra satisfies a few axioms that are not part of the definition of vertex algebra, but this is a relatively minor issue; as we point out in the introduction, there are many useful variants of the definition, depending on the context that one is interested in. It is the conceptual features of the notion of vertex (operator) algebra that are emphasized in the definitions in [B1] and [FLM6] that have led us to continue to use these two pieces of terminology as they have (usually) been used up to now. These conceptual features are discussed in the introduction and in the main text of the book. For instance, the two different terms reflect the issue of whether the algebras are viewed as "algebras of operators" or as algebras with a "multiplication operation" (or rather, an infinite sequence of multiplication operations); both viewpoints are "correct," and the best thing is to keep both in mind.

Primarily expository, this book does present a number of new results developed by the authors over the years in which drafts of this book were used in courses and seminars. These results are mentioned at the end of the introduction. In addition, in many places the proofs and treatments go beyond the originals, sometimes quite a bit.

The notion of vertex (operator) *super*algebra is a slight generalization of the notion of vertex (operator) algebra, and the basic general theory of vertex (operator) superalgebras is almost identical to the theory of vertex (operator) algebras; one needs to systematically keep track of $\pm$ signs at every stage. For simplicity of exposition, in this book we treat only vertex (operator) algebras. The original paper [Li3] *was* in fact written in the generality of vertex (operator) superalgebras, and the interested reader can consult [Li3] for the necessary (minor) changes.

The work on this book began when one author (J.L.) used the Japan lecture notes [Li2] by the other author (H.L.) in lectures presenting the work [Li3] for a graduate course at Rutgers University. H.L.'s work developed a theory of "representations" (in a specific sense) of a vertex operator algebra, and used it to construct families of examples of vertex operator algebras and modules; before [Li3], there had been no *general* approach available for constructing families of examples of vertex operator algebras and modules. Over a period of years, through many graduate courses and lectures by both authors, this book grew from informal lecture notes to a detailed self-contained introduction to the theory of vertex operator algebras, starting from the elementary fundamentals and leading to [Li3], and in fact, as it turned out, to a greatly expanded treatment of [Li3].

Actually, [Li3] was the published form of half of a 1993 preprint by H.L., and the other half of this 1993 preprint was published as [Li4]. H.L. is very grateful to Weiqiang Wang, then a graduate student at M.I.T., for his early interest in this preprint. H.L. would also like to thank Masahiko Miyamoto for organizing the wonderful workshop in the summer of 1994 at the Research Institute for Mathematical Sciences, Kyoto University, in which Chongying Dong, Yi-Zhi Huang and H.L. were invited to give lectures on vertex operator algebras. We are very grateful to many students and colleagues in lectures and seminars over the years, especially Chengming Bai, Corina Calinescu, Benjamin Doyon, Liang Kong, Machiel van Frankenhuysen and Lin Zhang. Their insightful comments and questions continually led us to seek increasingly better ways of illuminating critical and subtle points, and also of enhancing the readability by means of further explanatory remarks and cross-referencing. We thank Martin Karel for reading the manuscript, and Chongying Dong, Arne Meurman, Mirko Primc, and all the reviewers for their helpful comments. For the past two years, H.L. has used the text of this book in lectures at Harbin Normal University, Harbin, China. He would like to thank Shutao Chen, Yuwen Wang, Wende Liu, Shuqing Wang and the Mathematics Department for their hospitality, and to thank all the participants, Wende Liu, Shuqing Wang, Yinhua He, Li Li, Liqin Liu, Xuemei Liu and Xiuling Wang, for their interest and enthusiasm.

The formal calculus formulation of the theory of vertex operator algebras, including the Jacobi identity axiom, comes from [FLM6] and [FHL]. The "weak commutativity" and "weak associativity" properties come from [DL3]. J.L. would like to express his thanks to his collaborators on these works: Igor Frenkel, Arne Meurman, Yi-Zhi Huang and Chongying Dong.

The treatment in [DL3] actually included the vertex (operator) *super*algebra case, and in fact, this superalgebra case was itself a very special case of the full generality of [DL3]—that of "generalized vertex algebras" and "abelian intertwining algebras." Structures similar to generalized vertex algebras were also introduced and studied in [FFR] and [Mos1]. The special case of vertex (operator) superalgebras had already been studied in [Go1] and [Ts2].

It is a pleasure to thank Ann Kostant for encouraging us to publish this book with Birkhäuser and for her and her Birkhäuser colleagues' wonderful work in making the publication process so smooth.

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