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In Honor of Gerry Schwarz

H.E.A. Campbell
Aloysius G. Helminck
Hanspeter Kraft
David Wehlau
Editors

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Editors

H.E.A. Campbell
President
University of New Brunswick
Fredericton, NB E3B 5A3
Canada
eddy@unb.ca

Aloysius G. Helminck
North Carolina State University
Department of Mathematics
Raleigh, NC 27695
USA
loek@unity.ncsu.edu

Hanspeter Kraft
Universität Basel
Institut Mathematik
4051 Basel
Switzerland
Hanspeter.Kraft@unibas.ch

David Wehlau
Queen's University, Kingston
Department of Mathematics and Statistics
Kingston, ON K7L 3N6
Canada
wehlau-d@rmc.ca

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Foreword

I have been blessed with a wonderful group of teachers, students, colleagues, and collaborators during my mathematical career. I would have amounted to little without them. It was a special thrill for me that they organized a conference in celebration of my 60th birthday. I am honored by the wonderful group of participants, their talks, and their contributions to the proceedings.

I thank the gang of four who organized the party; H. E. A. (Eddy) Campbell, Loek Helminck, Hanspeter Kraft, and David Wehlau. We were all pleased when the Fields Institute agreed to host the event. The Fields' financial and logistical support as well as NSERC support through grants of Eddy and David made the presence of the many participants possible. My heartfelt thanks to all of them. Last, but not least, I want to thank Ann Kostant and Birkhäuser Boston for agreeing to publish the proceedings of the conference in the Progress in Mathematics series.

Boston, August 2008

Gerry Schwarz

*Dedicated to Gerry Schwarz on the occasion
of his 60th birthday, with affection from his
friends and admirers.*

Preface

Gerry Schwarz's many profound contributions to the study of algebraic groups, group actions and invariant theory have had a substantial effect on the progress of these branches of mathematics. This volume contains ten research articles contributed by his friends and colleagues as a tribute to him on the occasion of his sixtieth birthday.

The volume begins with a chapter by Brion. He studies actions of connected, not necessarily linear algebraic groups on normal algebraic varieties, and shows how to reduce them to actions of affine subgroup schemes.

Specifically, let G be a connected algebraic group acting on a normal algebraic variety X . Brion shows that X admits an open cover by G -stable quasi-projective varieties; this generalizes classical results of Sumihiro about actions of linear algebraic groups, and of Raynaud about actions on nonsingular varieties.

Next, if X is quasi-projective and the G -action is faithful, then the existence of a G -equivariant morphism $\psi : X \rightarrow G/H$ is shown, where H is an affine subgroup scheme of G , and G/H is an abelian variety. This generalizes a result of Nishi and Matsumura about actions on nonsingular varieties. If, in addition, X contains an open G -orbit, then G/H is the Albanese variety of X , and each fiber of ψ is a normal variety containing an open H -orbit. This reduces the structure of normal, almost homogeneous varieties to the case where the acting group is affine.

Finally, Brion presents a simple proof of an important result due to Sancho de Salas: any complete homogeneous variety X decomposes uniquely into a product $A \times Y$, where A is an abelian variety, and Y is a rational homogeneous variety. Moreover, we have the decomposition of connected automorphism groups $\text{Aut}^0(X) = A \times \text{Aut}^0(Y)$, and $\text{Aut}^0(Y)$ is semi-simple of adjoint type. This is an algebraic analogue of the Borel-Remmert structure theorem for compact homogeneous Kähler manifolds.

The chapter by Broer concerns rings of invariants of a finite group in arbitrary characteristic and the question of when such an invariant ring is a polynomial ring. The well-known theorem, due to Shephard and Todd and independently Chevalley, says that in the non-modular case (when the characteristic of the field $\mathbb{F}$ does not divide $|G|$) the ring of invariants $\mathbb{F}[V]^G$ is a polynomial ring if and only if the action

of G is generated by pseudo-reflections. Serre proved that the “only if” direction of this assertion holds in the modular case as well. The question of when the “if” direction holds is possibly the most important open question in modular invariant theory.

A representation V has the *direct summand property* if $\mathbb{F}[V]^G$ has a G -stable complement in $\mathbb{F}[V]$. In an earlier paper Broer conjectured that $\mathbb{F}[V]^G$ is a polynomial ring if and only if V has the direct summand property and the action of G is generated by pseudo-reflections. The “if” direction of this conjecture follows directly from Serre’s Theorem. In another earlier paper Broer showed that his conjecture is true under the additional hypothesis that the group G is abelian. Here Broer proves that if a representation V has the direct summand property then this property is inherited by the point stabilizers of every subspace of V . Using this he proves that his conjecture holds for irreducible G -actions.

The contribution of Daigle and Freudenburg concerns a famous conjecture made by Dolgachev and Weisfeiler in the 1970’s. In one formulation this conjecture states, that, if $\phi : \mathbb{A}^n \rightarrow \mathbb{A}^m$ is a flat morphism for which each fiber is $\mathbb{A}^{n-m}$ then ϕ is a trivial fibration. This paper gives a construction that unifies the examples given previously by Vénéreau and by Bhatwadekar and Dutta. It is still an open question as to whether the fibrations in these examples are trivial or not. Daigle and Freudenburg use locally nilpotent derivatives to give a systematic construction of a family of examples including the examples due to Vénéreau and to Bhatwadekar and Dutta. They show that many of the known results about these examples follow including the fact that these fibrations are stably trivial. This paper also includes an excellent summary of known results.

The chapter by Elmer and Fleischmann considers rings of invariants of finite modular groups. In the non-modular case, an important theorem due to Eagon and Hochster asserts that rings of invariants of finite groups are Cohen-Macaulay, i.e., that their depth is equal to their Krull dimension. It is well-known that this equality can (and usually does) fail for modular representations. In that setting, formulas for the depth of the ring of invariants are much sought after. Consider a modular representation of a p -group G defined over a field $\mathbb{F}$ of characteristic p . A lower bound on the depth is given by $\min\{\dim V, \dim V^P + \text{cc}(\mathbb{F}[V]) + 1\}$ where P is a p -Sylow subgroup of G and $\text{cc}(\mathbb{F}[V])$ denotes the cohomological connectivity of $\mathbb{F}[V]$. A representation which achieves this lower bound has been called *flat*. Elmer and Fleischmann here introduce a stronger notion *strongly flat* which implies flatness. They proceed to show that certain infinite families of representations of $C_p \times C_p$ are flat and so they have determined the depth of the corresponding rings of invariants. For the case $p = 2$, they are able to determine a complete classification of the strongly flat $C_2 \times C_2$ representations and to determine the depth of every indecomposable modular representation of $C_2 \times C_2$.

In their chapter Greb and Heinzner consider a Kähler manifold X with a Hamiltonian action of a compact Lie group K that extends to a holomorphic action of the complexification $K^{\mathbb{C}}$. The associated set of semistable points is open in X and has a reasonably well-behaved quotient, in particular there exists a complex space Q that parametrises closed $K^{\mathbb{C}}$ -orbits of semi-stable points. The group K acts on the

zero level set $\mathcal{M} := \mu^{-1}(0)$ of the given moment map $\mu : X \rightarrow \mathfrak{t}^*$, and the quotient $\mathcal{M}/K$ is homeomorphic to Q . This homeomorphism endows the complex space Q with a natural Kählerian structure that is smooth along the orbit type stratification. Since the quotient Q will in general be singular, the Kähler structure is locally given by continuous strictly plurisubharmonic functions. The quotient $\mathcal{M}/K$ is called the Kählerian reduction of X by K .

After summarizing the methods and results of the quotient theory outlined above, Greb and Heinzner show that this reduction process is natural in the sense that it can be formed in steps. This means that if L is a closed normal subgroup of K then the Kählerian reduction of X by L is a stratified Hamiltonian Kähler $K^{\mathbb{C}}/L^{\mathbb{C}}$ -space. Furthermore the Kählerian reduction of this space by K/L is naturally isomorphic to the Kählerian reduction of X by K .

The chapter by Helminck is directed to the study symmetric k -varieties, providing a survey of known results and giving some open problems. Consider a connected reductive algebraic group G defined over a field k of characteristic different from 2. Let $\theta : G \rightarrow G$ be an involution of G also defined over k . Let $H = G^{\theta}$ denote the points of G fixed by θ . Then H is a subgroup of G which is also defined over k . Write G_k (resp. H_k) to denote the k -rational points of G (resp. of H). Then the homogeneous space $X = G_k/H_k$ is what is called a symmetric k -variety. These are a generalization of both real reductive symmetric spaces and symmetric varieties and play an important role in many areas of mathematics, especially representation theory. Helminck considers the study of the orbits on X of parabolic subgroups of G_k . He gives a comprehensive survey of what is known including detailed discussions of the action of the Weyl group, the Bruhat order and the Richardson-Springer involution poset in this setting. He also considers the action on X by the fixed point group of another k -involution σ of G .

The chapter by Kostant develops a comprehensive theory of root systems for a complex semisimple Lie algebra $\mathfrak{g}$, relative to a non-trivial parabolic subalgebra $\mathfrak{q}$, generalizing the classical case, where $\mathfrak{q}$ is a Borel subalgebra. Let $\mathfrak{m}$ be Levi factor of such a $\mathfrak{q}$ and let $\mathfrak{t}$ be its centre: a nonzero element $v \in \mathfrak{t}^*$ is called a $\mathfrak{t}$ -root if the corresponding adjoint weight space $\mathfrak{g}_v$ is not zero. Some time ago, Kostant showed that $\mathfrak{g}_v$ is $\mathfrak{ad} \mathfrak{m}$ irreducible (and that all $\mathfrak{ad} \mathfrak{m}$ irreducibles are of this form). This result is the starting point for his generalization of the classical case, that is, the case when $\mathfrak{t}$ is a Cartan subalgebra. As an application, the author obtains new insight into the structure of the nilradical of $\mathfrak{q}$, and gives new proofs of the main results of Borel-de Siebenthal theory. This chapter contains many interesting and surprising insights into the structure of complex semisimple algebras, a classical subject, central to much of modern mathematics.

Kraft and Wallach study polarization and the nullcone. Suppose V is a complex representation of a reductive group G . Given an invariant $f \in \mathbb{C}[V]^G$, polarization is a classical technique used to generate some new invariants $Pf \subset \mathbb{C}[V^{\oplus k}]^G$ of the direct sum of k copies of V . The *nullcone* of a representation V is the subset $\mathcal{N} = \mathcal{N}_V$ of V on which all homogeneous non-constant invariants vanish. An important classical result, the Hilbert-Mumford criterion says that a point $v \in V$ lies in the nullcone if and only if there is a one-parameter subgroup which

annihilates v , i.e., if and only if there is a group homomorphism $\lambda^* : \mathbb{C}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t)v = 0$. Suppose that $f_1, f_2, \dots, f_n \in \mathbb{C}[V]^G$ and let m be a positive integer. Kraft and Wallach observe that the polarizations of $f_1, f_2, \dots, f_n$ will cut out the nullcones $\mathcal{N}_{V^{\oplus k}}$ for all $k \leq m$ if and only for every m dimensional subspace L contained in $\mathcal{N}_V$ there is a one-parameter subgroup λ^* of G which annihilates every point of L .

Specializing to the group $G = \mathrm{SL}_2(\mathbb{C})$ Kraft and Wallach show that if m is any positive integer, V is any representation of $\mathrm{SL}_2(\mathbb{C})$ and $f_1, f_2, \dots, f_n \in \mathbb{C}[V]^{\mathrm{SL}_2(\mathbb{C})}$ cut out the nullcone of V then the polarizations $Pf_1, Pf_2, \dots, Pf_n \in \mathbb{C}[V^{\oplus m}]^{\mathrm{SL}_2(\mathbb{C})}$ cut out the nullcone of $V^{\oplus m}$. This result is very surprising since most representations of a general group will not have this property.

They apply their results to study $Q_n := (\mathbb{C}^2)^{\otimes n}$ the tensor product of n copies of the defining representation of $\mathrm{SL}_2(\mathbb{C})$. This space, known as the space of n qubits, plays an important role in the theory of quantum computing.

Shank and Wehlau study the invariants of V_{p+1} , the $p+1$ dimensional indecomposable modular representation of the cyclic group G of order p^2 . Much work has been done in the past decade studying the modular invariant theory of the cyclic group of order p but this chapter is the first systematic study of the invariants of a modular representation of a cyclic group of higher order. Shank and Wehlau study the kG -module structure of $k[V_{p+1}]$, using a spectral sequence argument to obtain an explicit description of the ring of invariants. They also obtain a Hilbert series for all indecomposable G -representations of fixed type in $k[V_{p+1}]$ which they then combine to obtain an expression for the Hilbert series of the ring of invariants. This expression is surprisingly simple and compact.

Traves' chapter concerns rings of algebraic differential operators on quotient varieties such as the cone over a Grassmann variety, and actions of reductive groups on differential operators. The topic can be thought of as noncommutative invariant theory. The Weyl algebra of a complex vector space V is the ring $D(\mathbb{C}[V])$ of $\mathbb{C}$ -linear differential operators on $\mathbb{C}[V]$. More generally, if $R = \mathbb{C}[V]/I$ is the coordinate ring of an affine variety X then the ring of differential operators on X is

$$D(R) = D(\mathbb{C}[V]/I) = \frac{\{\theta \in D(\mathbb{C}[V]) : \theta(I) \subset I\}}{ID(\mathbb{C}[V])}.$$

An action of G on X induces an action on R but also on $D(R)$.

Traves considers the group $G = \mathrm{SL}_k(\mathbb{C})$ and its representation $V^{\oplus n}$ where V is its defining representation (of dimension k). He describes the two rings $D(R)^G$ and $D(R^G)$, through his use of subtle techniques developed by Schwarz.

The Fundamental Theorem of Invariant Theory gives a presentation of the ring of invariants $\mathbb{C}[G(k, n)]^{\mathrm{SL}_k(\mathbb{C})}$ where $G(k, n)$ is the affine cone over the Grassmanian of k planes in n space. In his book, *The Classical Groups. Their Invariants and Representations* Hermann Weyl suggested that the Fundamental Theorem should be extended to $D(R)^G$, giving a presentation of the invariant differential operators on the affine variety $G(k, n)$. Here Traves follows Weyl's suggestion by working with the graded algebra associated to $D(R)^G$. Applying the Fundamental Theorem to the

graded algebra, he obtains generators and relations. These then lift to generators of $D(R)^G$ and each of the relations on the graded algebra extends to a relation in $D(R)^G$ as well. He is also able to obtain Hilbert series for the graded algebras.

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Raleigh, NC, USA
Basel, Switzerland
Kingston, ON Canada
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H. E. A. Campbell
Loek Helminck
Hanspeter Kraft
David Wehlau

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Contents

Foreword	v
Preface	ix
Acknowledgements	xv
Some Basic Results on Actions of Nonaffine Algebraic Groups	1
Michel Brion	
1 Introduction	1
2 Proof of Theorem 1.1	5
3 Proof of Theorem 1.2	9
4 Proof of Theorem 1.3	11
5 Proof of Theorem 1.4	13
6 Examples	14
References	20
On Chevalley–Shephard–Todd’s Theorem in Positive Characteristic	21
Abraham Broer	
1 The direct summand property and point-stabilizers	22
2 Main result and tools for the proof	25
2.1 Tools	26
3 Details	28
3.1 Families	29
3.2 Exceptional cases	33
References	34
Families of Affine Fibrations	35
Daniel Daigle and Gene Freudenburg	
1 Introduction	35
2 Preliminaries	36
2.1 Notation and definitions	36
2.2 Some known results on affine fibrations	37

2.3	Locally nilpotent derivations	38
3	A criterion for affine fibrations	39
4	Dimension four	41
5	A remark on stable variables	42
	References	43
	On the Depth of Modular Invariant Rings for the Groups $C_p \times C_p$	45
	Jonathan Elmer and Peter Fleischmann	
1	Introduction	45
2	Flatness and strong flatness	47
3	A sufficient condition for strong flatness	49
4	Groups of the form $C_p \times C_p$	52
5	Decomposable representations	55
	References	60
	Kählerian Reduction in Steps	63
	Daniel Greb and Peter Heinzner	
1	Introduction	63
1.1	Reductive group actions on Kähler spaces	66
1.2	Stratifying holomorphic G -spaces	67
1.3	Kählerian reduction	68
2	Reduction in steps	71
2.1	Analytic reduction in steps	72
2.2	Kählerian reduction in steps	75
2.3	Stratifications in steps	77
3	Projectivity and Kählerian reduction	78
	References	81
	On Orbit Decompositions for Symmetric k-Varieties	83
	A. G. Helminck	
1	Introduction	83
2	Preliminaries about symmetric k -varieties	85
2.1	Notations	85
2.2	Classification of symmetric k -varieties	87
2.3	k -Anisotropic symmetric subgroups	88
3	Orbits of parabolic subgroups acting on symmetric varieties	90
3.1	k Algebraically closed and $P = B$ a Borel subgroup	91
3.2	k Algebraically closed and P a parabolic subgroup	92
3.3	Parabolic subgroups with σ -stable Levi subgroups	93
3.4	Some examples	94
3.5	$k \neq \bar{k}$ and P a minimal parabolic k -subgroup	96
4	Conjugacy classes of σ -stable tori	98
4.1	W -Action on V_A and V_k	98
4.2	Action of the Weyl group on twisted involutions	100
4.3	Conjugacy classes of σ -stable maximal k -split tori	101
4.4	H_k -Conjugacy classes in $\mathcal{A}_k^\sigma$ for $G = \mathrm{SL}(2, k)$	104

5	Bruhat order and twisted involution posets	105
5.1	Combinatorics of twisted involutions	107
5.2	Lifting σ' to an involution of G	110
5.3	σ -Stable parabolic subgroups	112
5.4	Relation between σ and σ'	113
5.5	W -Action on $\mathcal{I}_{\sigma'}$	114
5.6	Orbit closures	114
5.7	Bruhat order on V_A and V_k	116
5.8	Combinatorial Bruhat order on V_A	117
5.9	Admissible sequences in V	118
5.10	Bruhat order on $\mathcal{I}$ and $\mathcal{I}_{\sigma'}$	119
5.11	Combinatorics of the Richardson–Springer involution poset	122
6	Other orbit decompositions	123
6.1	Orbits of symmetric subgroups	123
6.2	Orbits of parahoric subgroups	124
	References	125
	Root Systems for Levi Factors and Borel–de Siebenthal Theory	129
	Bertram Kostant	
0	Introduction	129
0.1		129
1	Levi Factor Foot System	131
1.1		131
1.2		132
1.3		135
2	Properties of the $\mathfrak{t}$ -root System	136
2.1		136
2.2		138
2.3		140
2.4		141
3	Borel–de Siebenthal Theory, Special Elements, and the Lie Subalgebras they Define	144
3.1		144
3.2		146
3.3		147
3.4		148
4	Example	150
4.1		150
	References	152
	Polarizations and Nullcone of Representations of Reductive Groups	153
	Hanspeter Kraft and Nolan R. Wallach	
1	Linear subspaces of the nullcone	153
2	Some examples	155
3	General polarizations	158

4	Nullcone of several copies of binary forms	159
5	Generators and system of parameters for the invariants of 3-qubits	162
	References	166
Decomposing Symmetric Powers of Certain Modular Representations of Cyclic Groups 169		
R. James Shank and David L. Wehlau		
1	Introduction	169
2	Preliminaries	173
3	Computing $\mathbf{F}[V_{p+1}]^{\mathbf{Z}/p^2}$	176
4	The Noether number of V_{p+1}	183
5	Decomposing $\mathbf{F}[V_{p+1}]$	187
6	Generating functions	191
	References	195
Differential Operators on Grassmann Varieties 197		
Will Traves		
1	A fundamental theorem	199
2	The Hilbert series of $GrD(R)^G$ and $GrD(R^G)$	203
	References	206