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Vortices in Bose–Einstein Condensates

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Preface

Vortices are often associated with dramatic circumstances such as hurricanes; this type of vortex has been studied extensively in the framework of classical fluids. Nevertheless, its quantum counterpart has gained major interest in the past few years due to the experimental realization of Bose–Einstein condensates (BEC), a new state of matter predicted by Einstein in 1925. Vortices in BEC are quantized, and their size, origin, and significance are quite different from those in normal fluids since they exemplify "superfluid" properties.

Since the first experimental achievement of Bose–Einstein condensates in 1995 in alkali gases and the award of the Nobel Prize in Physics in 2001, the properties of these gaseous quantum fluids have been the focus of international interest in condensed matter physics. This book was both motivated by this intense activity, especially in the group of Jean Dalibard at the Ecole normale supérieure, but also by the constant development of mathematical techniques which could prove useful in tackling these problems, in particular in the group of Haim Brezis. This monograph is dedicated to the mathematical modelling of some specific experiments which display vortices and to a rigorous analysis of features emerging experimentally. It can serve as a reference for mathematical researchers and theoretical physicists interested in superfluidity and quantum condensates, and can also complement a graduate seminar in elliptic PDEs or modelling of physical experiments. There are two introductory chapters: the first is related to the physics background, while the second is devoted to the presentation of the mathematical results described in the book.

Vortices have been observed experimentally by rotating the trap holding the atoms in the condensate. In contrast to a classical fluid, for which the equilibrium velocity corresponds to solid body rotation, a quantum fluid such as a Bose–Einstein condensate can rotate only through the nucleation of quantized vortices beyond some critical velocity. There are two interesting regimes: one close to the critical velocity where there is only one vortex, and another at high rotation values, for which a dense lattice is observed. Another experiment consists of a superfluid flow around an obstacle: at low velocity, the flow is stationary; while at larger velocity, vortices are nucleated from the boundary of the obstacle.

One of the key issues is thus the existence of these quantized vortices. We address this issue mathematically and derive information on their shape, number, and location. In the dilute limit of the experiments, the condensate is well described by a mean field theory and a macroscopic wave function, solving the so-called Gross–Pitaevskii equation. The mathematical tools employed are energy estimates, Gamma convergence, and homogenization techniques. We prove existence of solutions which have properties consistent with the experimental observations. Open problems related to recent experiments are also presented. They will require the development of new tools related for instance, to microlocal analysis or time–dependent problems.

The suggestion for setting down these important ideas came from Haim Brezis, and I would like to thank him warmly for his constant enthusiasm and support while I was working on it. Many tools used here have been developed by either him or his school. I am glad to be able to present an application of this beautiful mathematics to today's physics.

I am also extremely grateful to Jean Dalibard, who has always been willing to take time to share his experiments, his ideas, and his interests in how mathematics can contribute to physics. Working together with him and writing a joint paper was a real pleasure and a source of mathematical problems for many years to come. I would also like to thank him for his careful reading of this manuscript. Before working with Jean, I had the opportunity of many fruitful discussions with members of his group, in particular Vincent Bretin, Yvan Castin, and David Guéry-Odelin. I have always appreciated their open minds and interest in mathematics. I would like to thank David in particular for his comments leading to improvements in the introductory parts of this book.

I owe my personal interest in interdisciplinary topics to the joint efforts of Etienne Guyon, Yves Pomeau, and Henri Berestycki, who launched a program for students at the Ecole normale supérieure to spark interest in problems on the border between mathematics and physics. This effort was a real success, as were the various maths-physics meetings in Foljuif, a property of the Ecole normale supérieure. At one of them, I met Yvan Castin and realized that we had mathematical tools that could help in understanding problems emerging in rotating Bose–Einstein condensates. I would like to again my deep gratitude to Etienne Guyon, Yves Pomeau, and my supervisor Henri Berestycki, for all that I discovered has been thanks to them.

I would like to also thank, of course, all my collaborators on these topics, in particular: Tristan Rivière, with whom this huge program started and the evidence of vortex bending occurred; Bob Jerrard, whose involvement in understanding the shape of vortices was quite influential and with whom it was a pleasure to work in Vancouver, Milan, Istanbul, and Minneapolis (I thank all the hosting institutions) and who has undertaken a very careful reading of the book; Qiang Du and Ionut Danaila, who have performed, on different topics, beautiful numerical computations; Stan Alama and Lia Bronsard, who came to Paris and became interested in these topics when they discovered them; Xavier Blanc, with whom I am very happy to have worked with on many projects; and very recently Francis Nier, who has allowed me to discover microlocal analysis and Bargmann transforms, which have proved to

be quite useful in tackling these problems. It was also very rewarding to work with outstanding physicists, Jean Dalibard and Yves Pomeau. Many colleagues all over the world have mentioned that I was quite lucky to have had this opportunity and to have found a common language to speak. I certainly believe it.

Part of this monograph was taught as a Ph.D. course at Paris 6 in 2003–2004. One of the results presented here was obtained by two of these Ph.D. students, Radu Ignat and Vincent Millot, whom I jointly supervised with Haim Brezis. I am pleased to describe their work in one of the chapters.

The quality of the presentation of this book was greatly improved thanks to all the lectures given in various universities or summer schools. I would like to thank in particular: Luis Caffarelli and Irene Gamba, Peter Constantin, Peter Sternberg, Miguel Escobedo, Gero Friesecke, Fang-Hua Lin, Stefan Muller, Tristan Rivière, and Juan-Luis Vazquez. I have also benefited from informal discussions with Fabrice Bethuel, Petru Mironescu, Sylvia Serfaty, Etienne Sandier, and Didier Smets.

I take the opportunity here to express my gratitude to all my colleagues in the Laboratoire Jacques-Louis Lions, in particular: to Yvon Maday, who is a very enthusiastic head of department; to my office mate Xavier Blanc; to my office neighbours, Edwige Godlewski and Francois Murat; to Olivier Glass, Frédéric Hecht, Simon Masnou, and the staff in the laboratory, Danielle Boulic, Michel Legendre, Jacques Portes, and Liliane Ruprecht.

The writing of this book was made possible by my position at CNRS, which should be naturally associated with the outcome of such interdisciplinary effort. My research during this period was supported by a CNRS grant for young researchers and a French ministry of research grant, ACI "Nouvelles interfaces des mathématiques." Some of the open problems were derived during a maths–physics meeting organized with David Guéry-Odelin at the "Fondation des Treilles" in Tourtour, and I would like to acknowledge their welcome.

Finally, I would like to thank my family and friends for their constant support in the preparation of the manuscript and all the people at Birkhäuser, in particular Ann Kostant, for their help.

Paris, October 31, 2005, *Amandine Aftalion*

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