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Compactifications of Symmetric and Locally Symmetric Spaces

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To

Gaby Borel, Lan Wang

&

Lena, Emily, and Karen

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Preface

Symmetric spaces and locally symmetric spaces occur naturally in many branches of mathematics as moduli spaces and special manifolds, and are often noncompact. Motivated by different applications, compactifications of symmetric spaces and locally symmetric spaces have been studied intensively by various methods. For example, a typical method to compactify a symmetric space is to embed it into a compact space and take the closure, while a typical method to compactify a locally symmetric space is to attach ideal boundary points or boundary components. In this book, we give uniform constructions of most known compactifications of both symmetric and locally symmetric spaces together with some new compactifications. We also explain how different types of compactifications arise and are used; in particular, why there are so many different kinds of compactifications of one space, and how they are related to each other. We hope to present a comprehensive survey of compactifications of both symmetric and locally symmetric spaces. It should be pointed out that this book emphasizes the geometric and topological aspects of the compactifications; on the other hand, we have tried to provide adequate references to omitted topics.

The book is divided into three parts corresponding to different classes of compactifications. In Part I, we study compactifications of Riemannian symmetric spaces in the usual sense that the symmetric spaces are open and dense subsets. In Part II, we study compact smooth manifolds in which the disjoint union of more than one but finitely many symmetric spaces is contained as an open, dense subset, and the closure of each copy of the symmetric spaces is a manifold with corners. In Part III, we study compactifications of locally symmetric spaces, and their relations to the metric and spectral properties of the locally symmetric spaces.

Though these parts treat different types of compactifications, they are closely related in various ways. In fact, there are several basic, unifying themes in this book:

1. Siegel sets of rational parabolic subgroups and the reduction theory of arithmetic groups have played an important role in the study of compactifications of locally symmetric spaces. We show that compactifications of a symmetric space X can also be studied through a generalization for real parabolic subgroups of

Siegel sets of rational parabolic subgroups and the reduction theory of arithmetic groups. Therefore, compactifications of symmetric and locally symmetric spaces can be studied in parallel using similar methods.

2. Compactifications of a locally symmetric space $\Gamma\backslash X$ can be studied and understood better through compactifications of the homogeneous space $\Gamma\backslash G$, which is a principal fiber bundle over $\Gamma\backslash X$ whose fibers are maximal compact subgroups of G.
3. Certain compactifications of symmetric spaces in Part II can be realized by gluing other compactifications from Part I in a manner analogous to the method of obtaining a closed manifold by doubling a manifold with boundary, which is called self-gluing in this book. In Part II, we describe a new and simpler technique for constructing analytic structures on compactifications by complexifying the symmetric spaces into complex symmetric varieties and using compactifications of the complex symmetric varieties as projective varieties.
4. Although compactifications of locally symmetric spaces are traditionally constructed using compactifications of symmetric spaces, we construct in Part III these compactifications independently of compactifications of symmetric spaces, and hence use only the $\mathbb{Q}$-structure of the spaces.
5. In Part III, we consider metric properties of compactifications of locally symmetric spaces. This gives a new perspective on sizes of compactifications and deepens our understanding of their properties. It also simplifies applications to extension properties of holomorphic maps from the punctured disk to Hermitian locally symmetric spaces. Furthermore, it clarifies relations to the continuous spectrum of the locally symmetric spaces.

The basic plan of the book was worked out and agreed upon by the two authors around December 2002. Part II was mostly written by the first author before his unexpected death on August 11, 2003.[1] The task of finishing this book fell to the second author, who regrets that his style may fail to meet the high standards of the first author, but who must nevertheless take responsibility for any errors or inaccuracies in the text.

This book is partially based on the joint papers [BJ1] [BJ2] [BJ3] [BJ4]. The book project was proposed by N. Wallach, an editor of this series, to the authors near the end of the European summer school on Lie theory in Marseille-Luminy, France, 2001, where the authors gave a joint series of lectures on compactifications of symmetric spaces and locally symmetric spaces.

The second author would like to thank N. Wallach for the book proposal and comments on an earlier version of this book, and J.P. Anker and P. Torasso for inviting the authors to give the lectures at the European summer school on Lie theory in 2001. He would also like to thank N. Mok for arranging and for inviting him to participate in the multiyear program on Lie theory organized by the first author at the University of Hong Kong, where some of the joint work was carried out, and

[1] Except for the introductions to Part II and the chapters there, several comments, and minor changes of notation for consistency, Part II was the version the first author finished and in the form he liked in June 2003.

his teacher M. Goresky for providing the carefully taken notes [Mac] that motivated Chapter 12 in Part III and for many helpful and encouraging conversations on various topics, comments, and suggestions. He thanks G. Prasad for helpful conversations and comments, and S. Zucker and A. Korányi for very helpful correspondence, conversations, specific and general suggestions, and detailed comments on an earlier version of the book. The second author would also like to thank his teacher S.T. Yau for inviting both authors to run a multiyear summer school on *Lie groups and automorphic forms* at the Center of Mathematical Sciences at Zhejiang University, which further encouraged us to work on this book project. It is sad that the first author could not attend any of the activities he planned at the center in Hangzhou.[2]

The second author would also like to thank R. Lazarsfeld for suggestions about the index and the layout of the book, E. Gustafsson, the first author's secretary, for typing up Part II, and A. Kostant for help and many suggestions during the preparation of this book. A seminar talk by W. Fulton on how to write mathematics led to some improvements of the writing and style of this book. The work of this book has been partially supported by NSF grants and an A.P. Sloan research fellowship.

Finally, the second author would like to thank his wife, Lan Wang, and his three daughters, Lena, Emily, and Karen, for their support, understanding, and patience during the intense writing and revising periods of this book.

[2] The second author co-organized an international conference titled *Algebraic groups, arithmetic groups, representation theory and automorphic forms* in memory of the first author at the Center of Mathematical Sciences at Zhejiang University, Hangzhou, July 26–30, 2004. The second author would like to thank the director of the center, S.T. Yau, and its staff members for their efforts and hard work to make the conference a success.

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