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Introduction to Nonlinear Dispersive Equations

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Preface

The goal of this monograph is to present an introduction to a sampling of ideas and methods from the subject of nonlinear dispersive equations. This subject has been of great interest and has rapidly developed in the last few years. Here we will try to expose some aspects of the recent developments.

The presentation is intended to be self-contained, but we will assume that the reader has knowledge of the material usually taught in courses of theory of one complex variable and integration theory.

This monograph is the product of lecture notes used for mini-courses and graduate courses taught by the authors. The first version of the lecture notes were written by Gustavo Ponce with Wilfredo Urbina from the Universidad Central de Venezuela and designed to teach a mini-course at the Venezuelan School of Mathematics in Mérida, Venezuela, in 1990. A second version of those notes was presented by Gustavo Ponce at the Colombian School of Mathematics in Cali, Colombia in 1991. These notes comprise a part of the materials covered in the first six chapters of the present monograph. Most of the original notes were used to teach various graduate courses at IMPA and UNICAMP by Felipe Linares. During these lectures the previous versions were complemented with some new materials presented here. These notes were also used by Hebe Biagioni and Marcia Scialom from UNICAMP in their seminars and graduate courses. The idea to write the present monograph arose from the need for a more complete treatment of these topics for graduate students.

Before going any further we would like first to give a notion of what a partial differential equation of dispersive type is. We will do this in the one-dimensional frame. We consider a linear partial differential equation

$$F(\partial_x, \partial_t)u(x,t) = 0, \tag{0.1}$$

where F is a polynomial in the partial derivatives. We look for plane wave solutions of the form $u(x,t) = A\, e^{i(kx-\omega t)}$ where A, k, and ω are constants representing the amplitude, the wavenumber, and the frequency, respectively. Hence u will be a solution if and only if

$$F(ik, -i\omega) = 0. \tag{0.2}$$

This equation is called the *dispersion relation*. This relation characterizes the plane wave motion. In several models we can write ω as a real function of k, namely,

$$\omega = \omega(k).$$

The phase and group velocities of the waves are defined by

$$c_p(k) = \frac{\omega}{k} \qquad \text{and} \qquad c_g = \frac{d\omega}{dk}.$$

The waves are called *dispersive* if the group velocity $c_g = \omega'(k)$ is not constant, i.e., $\omega''(k) \neq 0$. In the physical context this means that when time evolves the different waves disperse in the medium, with the result that a single hump breaks into wave-trains.

To present the material we have chosen to study two very well known models in the class of nonlinear dispersive equations: the Korteweg–de Vries equation

$$\partial_t v + \partial_x^3 v + v\partial_x v = 0, \tag{0.3}$$

where v is a real-valued function and the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u, \bar{u}), \tag{0.4}$$

where u is a complex-valued function.

Before commenting on the theory presented in this monograph regarding these equations we would like to say few words concerning the physical models described by these equations in the context of water waves.

The first model (0.3) goes back to the discovery of Scott Russell in 1835 of what he called a traveling wave. This equation describes the propagation of waves in shallow water and was proposed by Diederik Johannes Korteweg and Gustav de Vries in 1895 [KdV]. In the one-dimensional context the (cubic) nonlinear Schrödinger equation (0.4) with $f(u, \bar{u}) = |u|^2 u$ models the propagation of wave packets in the theory of water waves.

We also have to mention that there is a very well known strong relationship between these two equations and the theory of completely integrable systems, or *Soliton theory*.

In many cases, we present the details of simple proof, which may not be that of the strongest result. We give several examples to illustrate the theory. At the end of every chapter we complement the theory described either with a set of exercises or with a section with comments on open problems, extensions, and recent developments.

The first three chapters attempt to review several topics in Fourier analysis and partial differential equations. These are the elementary tools needed to develop the theory in the rest of the notes.

The properties of solutions to the linear problem associated to the Schrödinger equation are discussed in Chapter 4. Then the initial value problem associated to (0.4) and properties of its solutions are studied in Chapters 5 and 6. Chapters 7 and

8 are devoted to the study of the initial value problem for the generalized Korteweg–de Vries equation. A survey of results concerning several nonlinear dispersive equations that generalize (0.3) and (0.4) as Davey–Stewartson systems, Ishimori equations, Kadomtsev–Petviashvili equations, Benjamin–Ono equations, and Zakharov systems is presented in Chapter 9. In the last chapter we present the most recent result regarding local well-posedness for the nonlinear Schrödinger equation.

We shall point out that by no means our presentation is completely exhaustive. We refer the reader to the lecture notes by Cazenave [Cz1], [Cz2] and the books by Sulem and Sulem [SS2], Bourgain [Bo2], and Tao [To7]. In these works many topics not covered in these notes are studied in detail.

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