Chapter 1
Introduction

This book is devoted to the development of a calculus of variations that can apply to a large number of partial differential systems and evolution equations, many of which do not fit in the classical Euler-Lagrange framework. Indeed, the solutions of many equations involving nonlinear, nonlocal, or even linear but non self-adjoint operators are not normally characterized as critical points of functionals of the form $\int_\Omega F(x,u(x),\nabla u(x))dx$. Examples include *transport equations* on a smooth domain Ω of $\mathbf{R}^n$ such as

$$\begin{cases} \Sigma_{i=1}^n a_i \frac{\partial u}{\partial x_i} + a_0 u = |u|^{p-1}u + f & \text{on} \quad \Omega \subset \mathbf{R}^n, \\ u(x) = 0 & \text{on} \quad \Sigma_-, \end{cases} \tag{1.1}$$

where $\mathbf{a} = (a_i)_i : \overline{\Omega} \to \mathbf{R}^n$ is a given vector field, $p > 1$, $f \in L^2(\Omega)$, and Σ_- is the entrance set $\Sigma_- = \{x \in \partial\Omega; \mathbf{a}(x)\cdot\mathbf{n}(x) < 0\}$, $\mathbf{n}$ being the outer normal on $\partial\Omega$.

Another example is the equation

$$\begin{cases} \operatorname{div}(T(\nabla f(x))) = g(x) & \text{on} \quad \Omega \subset \mathbf{R}^n, \\ f(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where T is a monotone vector field on $\mathbf{R}^n$ that is not derived from a potential.

Similarly, dissipative initial-value problems such as the *heat equation*, *porous media*, or the *Navier-Stokes evolution*

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u + f = \alpha\Delta u - \nabla p & \text{on } \Omega \subset \mathbf{R}^n, \\ \operatorname{div} u = 0 & \text{on } [0,T]\times\Omega, \\ u = 0 & \text{on } [0,T]\times\partial\Omega, \end{cases} \tag{1.3}$$

where $\alpha > 0$ and $f \in L^2([0,T]\times\Omega)$, cannot be solved by the standard methods of the calculus of variations since they are not Euler-Lagrange equations of action functionals of the form $\int_0^T L(t,x(t),\dot{x}(t))dt$. Our goal here is to describe how these examples and many others can still be formulated and resolved variationally by means of a *self-dual variational calculus* that we develop herein.

The genesis of our approach can be traced to physicists who have managed to formulate – if not solve – variationally many of the basic nonself-adjoint equations of quantum field theory by minimizing their associated action functionals. Indeed,

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