Part III
SELF-DUAL SYSTEMS AND THEIR ANTISYMMETRIC HAMILTONIANS

Stationary Navier-Stokes equations, finite-dimensional Hamiltonian systems, and equations driven by nonlocal operators such as the Choquard-Pekar equation are not completely self-dual systems but can be written in the form

$$0 \in \Lambda u + \overline{\partial} L(u),$$

where L is a self-dual Lagrangian on $X \times X^*$ and $\Lambda : D(\Lambda) \subset X \to X^*$ is a, not necessarily linear, operator. They can be solved by minimizing the functionals

$$I(u) = L(u, -\Lambda u) + \langle \Lambda u, u \rangle$$

on X by showing that their infimum is zero and that it is attained. These functionals are typical examples of a class of *self-dual functionals* that we introduce and study in this part of the book. They are defined as functionals of the form

$$I(u) = \sup_{v \in X} M(u, v),$$

where M is an *antisymmetric Hamiltonian* on $X \times X$, which contain and extend in a nonconvex way, the Hamiltonians associated to self-dual Lagrangians by standard Legendre duality (i.e., in one of the variables).

The class of antisymmetric Hamiltonians is quite large and easier to handle than the class of self-dual Lagrangians. It contains “Maxwellian” Hamiltonians of the form $M(x,y) = \varphi(y) - \varphi(x)$, with φ being convex and lower semicontinuous, but also those of the form $M(x,y) = \langle \Lambda x, x - y \rangle$, where Λ is a suitable – possibly nonlinear – operator, as well as their sum.

Self-dual functionals turn out to have many of the variational properties of the completely self-dual functionals, yet they are much more encompassing since they allow for the variational resolution of a larger class of linear and nonlinear partial differential equations.