

**Part III**  
**SELF-DUAL SYSTEMS AND THEIR**  
**ANTISYMMETRIC HAMILTONIANS**

Stationary Navier-Stokes equations, finite-dimensional Hamiltonian systems, and equations driven by nonlocal operators such as the Choquard-Pekar equation are not completely self-dual systems but can be written in the form

$$0 \in \Lambda u + \bar{\partial}L(u),$$

where  $L$  is a self-dual Lagrangian on  $X \times X^*$  and  $\Lambda : D(\Lambda) \subset X \rightarrow X^*$  is a, not necessarily linear, operator. They can be solved by minimizing the functionals

$$I(u) = L(u, -\Lambda u) + \langle \Lambda u, u \rangle$$

on  $X$  by showing that their infimum is zero and that it is attained. These functionals are typical examples of a class of *self-dual functionals* that we introduce and study in this part of the book. They are defined as functionals of the form

$$I(u) = \sup_{v \in X} M(u, v),$$

where  $M$  is an *antisymmetric Hamiltonian* on  $X \times X$ , which contain and extend in a nonconvex way, the Hamiltonians associated to self-dual Lagrangians by standard Legendre duality (i.e., in one of the variables).

The class of antisymmetric Hamiltonians is quite large and easier to handle than the class of self-dual Lagrangians. It contains “Maxwellian” Hamiltonians of the form  $M(x, y) = \varphi(y) - \varphi(x)$ , with  $\varphi$  being convex and lower semicontinuous, but also those of the form  $M(x, y) = \langle \Lambda x, x - y \rangle$ , where  $\Lambda$  is a suitable – possibly nonlinear – operator, as well as their sum.

Self-dual functionals turn out to have many of the variational properties of the completely self-dual functionals, yet they are much more encompassing since they allow for the variational resolution of a larger class of linear and nonlinear partial differential equations.