Instability in Models Connected with Fluid Flows I

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Instability in Models Connected with Fluid Flows I

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Two volumes of the *International Mathematical Series* present various topics on control theory, free boundary problems, the Navier–Stokes equations, attractors, first order linear and nonlinear equations, partial differential equations of fluid mechanics, etc. with the focus on the key question in the study of mathematical models simulating physical processes:

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  Efim Dinaburg and Yakov Sinai, Vol. I

  — The analyticity of periodic solutions of the 2D Boussinesq system.
  Maxim Arnold, Vol. I

  — Navier–Stokes equations in cylindrical domains. Leray approximations, Leray–Navier–Stokes equations, the Helmholtz projector and stationary Stokes problem, the classical Navier–Stokes problem.
  Sergey Zelik, Vol. II

- **First order linear and nonlinear equations**

  — Nonlinear dynamics of a system of particle-like wavepackets, reduction of wavepacket interaction systems to averaged ones, superposition principle and decoupling of wavepacket interaction systems.
  Anatoli Babin and Alexander Figotin, Vol. I

  — Transport equations with discontinuous coefficients, Keyfitz-Kranzer type hyperbolic systems, generalized solutions of the Cauchy problem, existence, uniqueness, and renormalization property.
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— Controllability and accessibility of equations of dynamics of incompressible fluids controlled by low-dimensional (degenerate) forcing, controllability of Navier–Stokes / Euler equations on a two-dimensional sphere and on a generic Riemannian surface.
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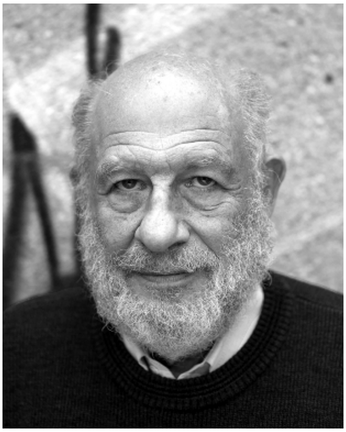

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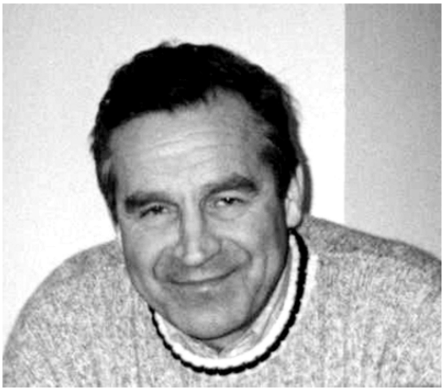

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Preface

1. Overview

These two volumes are devoted to mathematical analysis of equations of continuous media (mostly fluids) describing phenomena for which the basic underlying physics, i.e., their relation with First Principles, is well understood and broadly accepted. One of the most important mathematical issues is how these equations can be used for an accurate description of "matter." At present, this question is especially urgent in virtue of at least three interconnected factors: new engineering problems, advantages of functional analysis, and the emergence of digital computing.

- Modern engineering problems involve physics at different levels of accuracy, corresponding to different equations. The properties of these equations and the relations between them turn out to be important for applications.

  For instance, the Navier–Stokes equations and the Maxwell equations are the most commonly used to compute quantities related to fluids and electromagnetic waves respectively. However, if a medium is rarefied, other (more refined) equations should be used. This is typically the case for the re-entry in the atmosphere of a space vehicle transiting very rapidly from a region where the gas is rarefied to a region of gas with normal density. Then the Boltzmann equation should be used.

  In the same way, the use of the transport kinetic equation is imperative for devices so small that the flux of electrons cannot reach thermal equilibrium. At the other end of the scale spectrum, one confronts issues like climate evolution, and therefore it is necessary to use equations describing the interaction between the ocean and the atmosphere or the stability of very large structures in fluids such as anticyclones and the Jupiter red spot.

- During the evolution of mathematics from the 19th to the 20th century, the emphasis in studying these equations shifted from trying to find an explicit form of solutions to investigating equations by functional analysis methods due to Hilbert, Banach, and others.

- In fact, the systematic use of functional analysis is naturally combined with access to digital computing, also not relying on explicit solutions. Functional analysis is of paramount importance not only for computing error estimates between a real solution and its discrete approximation, but also, most significantly, for constructing a discrete version of the problem that retains the basic properties of the original problem (a necessary condition for convergence). For instance, in fluid mechanics, any discrete approximation should preserve mass, momentum, and energy. As predicted by von Neumann in 1946, digital computation provides information not available through other methods. It is important to note that, combined with mathematical analysis, these computations have led to mathematical discoveries. The most classical examples involve dynamical systems.

  i) The observation of the singular behavior of a discrete version of the Kortweg-de Vries equation made in 1955 by Fermi, Pasta, and Ulam [**4**], which led Lax, in 1968, to the study of the integrability of the Kortweg-de Vries equation by using the so-called Lax pair [**8**].

  ii) The discovery of *strange attractors* by Lorentz [**10**] and Hénon [**6**] on the basis of numerical experiments, which motivated a systematic research on properties of attractors; for fluids, in particular, starting with the contribution of Ladyzhenskaya [**7**] in 1972.

While the range of applications of partial differential equations is extremely large, from quantum theory to biology, the equations of fluid mechanics have a particular status. It turns out that success in the investigation of these equations leads to new results in many other nonlinear problems. Therefore, the equations of fluid mechanics often serve as models in the study of other nonlinear problems arising in applications and as a constant stimulus for new mathematical discoveries.

A striking example is the notion of a weak solution, implicitly presented in the analysis of shocks in conservation laws obeying the *Rankine–Hugoniot condition.* This notion was formalized for the construction of turbulent solutions to the Navier-Stokes equations by Leray [**9**] in 1933 and was ultimately completed with the creation of distribution theory by Sobolev [**16, 17**] in 1935/36 and by Schwartz [**15**] in 1945.

A description of a physical process by PDEs can be adequate only if a certain stability property interpreted depending on the physical problem takes place.

For linear partial differential equations the first formal definition of stability (well-posedness) was given by Hadamard [**5**] in 1904. In 1937, based on the notion of stability in the sense of Hadamard, Petrowsky [**13**] proposed a systematic classification of general systems of PDEs.

The nonlinear structure of equations describing fluid flows dictates different approaches to the introduction of the notion of stability. In addition to the classical stability (well-posedness in the Hadamard sense), there are various definitions of stability reflecting specific mathematical aspects of physical problems. In particular, the following variants will be discussed in these volumes:

- the large time behavior of solutions, which is related to the Lyapunov stability of stationary solutions and attractors
- stability relative to initial data (for example, wave packets)
- stability of averaged models obtained by introducing an infinite-dimensional measure driven by a stochastic process
- stability of free-boundary problems
- stability problems in control theory

2. Classification of Contributions and Comments

The idea was to gather a collection of contributions from experts to cover current approaches to the study of stability of mathematical models simulating processes in fluid flows. We present several directions in this area that are different by methods and problem statements, but all of them are joined by the final goal of research: to clarify whether the mathematical model under consideration possesses the property of stability (instability) in a certain interpretation of this notion.

Below we classify the papers in both volumes according to the selected directions and give our comments on presented results.

2.1. Navier-Stokes equations. General results (existence and smoothness of solutions).

This direction is presented by three papers, where nontrivial situations are considered; in particular, the problem can be stated in an unbounded domain or the solution can be of infinite energy.

[DS] **Efim Dinaburg and Yakov Sinai**, *Existence theorems for the 3D Navier–Stokes system having as initial conditions sums of plane waves*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 289–300.

In this paper, the existence theorem for the Cauchy problem for the 3D Navier-Stokes equations is proved in the case, where the initial condition is a finite sum of plane waves. The time interval, where the solution exists, depends on the initial condition. We emphasize that the initial condition is not assumed to be of finite energy. The proof is based on the method of power series which is of independent interest. There is also an example, where a solution exists on a time interval independent of the initial condition. We should note that the existence of solutions on an arbitrary time interval was earlier obtained by another method in [**18**] for almost all coefficients of the initial quasiperiodic polynomial with respect to the Lebesgue measure.

[A] **Maxim Arnold**, *Analyticity of periodic solutions of the 2D Boussinesq system*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 37–52.

The paper by Sinai's former student M. D. Arnold is devoted to the proof of the analyticity of periodic solutions to the 2D Boussinesq system, an extension of the Navier-Stokes equations, and uses the method of [**11**].

[Ze] **Sergey Zelik**, *Weak spatially nondecaying solutions of 3D Navier–Stokes equations in cylindrical domains*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 329–376.

Zelik develops an infinite energy theory for the Navier–Stokes equations in unbounded $3D$ cylindrical domains. Based on this theory, he establishes the existence of a weak solution in a uniformly local phase space (without any spatial decay assumptions), the dissipativity of the solution, and the existence of the so-called trajectory attractor. In particular, this phase space contains the 3D Poiseuille flows. Estimates on the size of the attractor in terms of the kinematic viscosity are also obtained.

2.2. First order linear and nonlinear equations.

The difference in statements and approaches presented in the papers of this direction reflects the rich variety of subjects and methods in current investigations of different aspects of stability (instability) in this area.

[BF] **Anatoli Babin and Alexander Figotin**, *Nonlinear dynamics of a system of particle-like wavepackets*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 53–134.

The authors highlight the propagation properties of quasilinear hyperbolic equations by introducing a special class of the so-called *particle-like wave packets*. This notion has a dual nature. On one hand, a particle-like wave packet is a wave with a well-defined principal wave vector. On the other hand, it is a particle in the sense that it can be assigned to a well-defined position in space. As was established in this paper, under this nonlinear evolution, a generic multi-particle wave packet remains a multi-particle wave packet with high accuracy and the constituent single particle-like wave packet not only preserves the principal wave number, but also has a well-defined space position evolving with constant velocity (their group velocity). To prove these results, the authors use properties of the linear (hyperbolic) part of the system under consideration and the particle-like wave packet structure of the initial data. The methods used in [BF] are close to those of [Ch] and [GMN].

[P] **Evgenii Panov**, *Generalized solutions of the Cauchy problem for a transport equation with discontinuous coefficients*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 23–84.

Transport equations with discontinuous coefficients arise in the analysis of various nonlinear systems of conservation and balance laws with linear degeneracy of some components. For example, the system of Keyfitz–Kranzer type, known in magnetohydrodynamics, reduces to a system of such a kind. Furthermore, as is known [**12**], transport equations with discontinuous coefficients appear as the adjoint equations corresponding to hyperbolic systems of conservation laws. Panov presents the well-posedness theory for general nonhomogeneous transport equations which can be applied for establishing the existence and uniqueness of strong entropy solutions to the Cauchy problem for Keyfitz–Kranzer type systems.

[R] **Evgenii Radkevich**, *Irreducible Chapman–Enskog Projections and Navier–Stokes approximations*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 85–154.

In order to derive the viscosity and heat diffusion coefficients from the Boltzmann equation, Chapman and Enskog proposed an approximation of solutions to the Boltzmann equation in terms of macroscopic quantities or *moments* of the solution. This approach works very well for the first-order approximation with respect to the Knudsen number ε. This leads to the compressible Navier–Stokes equation and provides a way to derive the viscosity and heat diffusion coefficients from First Principles. For the next order in ε, the Burnett equation appears, an ill-posed equation in the sense of Hadamard. As was noted in [**2**], a very good model for relaxation to the equilibrium property of the Boltzmann equation is the nonlinear Euler equation with relaxation term of order ε^{-1}. Based on spectral analysis, Radkewich proposed some other derivation. In particular, he proved that, in the case of an odd number of equations, a well-posed approximation of dependent variables of any order can be expressed as an equation of one variable. If the number of equations is even, the approximation can be expressed via two macroscopic variables.

2.3. Finite time instabilities of 3*d* incompressible Euler equations.

The question whether solutions to the 3d incompressible Euler equations with finite energy and smooth initial data may blow up in finite time is still open. However, it is known that a family of smooth initial data may generate growth in the vorticity that, even if not infinite, may be arbitrarily large. Furthermore, even in the 2d case, a family of initial data with nonuniformly bounded vorticity may generate pathological behavior. In [Ch] and [GMN], the reasons leading to such patologies are investigated.

[Ch] **Christophe Cheverry**, *Recent results in large amplitude monophase nonlinear geometric optics*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 267–288.

Using methods of nonlinear geometric optics applied to a family of oscillating initial data, Cheverry shows that the weak limit of the corresponding solutions does not satisfy the Euler equation any more.

[GMN] **Francois Golse, Alex Mahalov, and Basil Nicolaenko**, *Bursting dynamics of the* 3*D Euler equations in cylindrical domains*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 301–338.

To exhibit the stabilizing effect of a fast rotation, the authors consider solutions to the Euler equations in a finite cylinder with initial data that is a bounded perturbation of a large uniform rotation Ω along the axis of the cylinder. Conjugating the solution with the Poincaré–Steklov operator (the rotation in the space of divergence-free functions), they construct a resonant limit system. Special solutions (in particular, periodic and integrable ones) are studied by methods of the classical Hamiltonian mechanics for rigid bodies. Using a shadowing lemma, the authors find that the solutions to the original Euler equation have similar behavior. From the Editors' point of view, the major and remarkable result is the construction of time periodic solutions with large variation of the ratio of the $H^s(t)$ norms between two different times t_1 and t_2 (for any s). Such a *bursting* dynamics, without singularities, corresponds to the so-called *depletion* in the study of the Euler equations.

2.4. Large time asymptotics of solutions.

The analysis of the large time behavior of solutions to the fluid equations covers many applications and is connected with basic physical issues, for instance, the route to turbulence. At the same time, it can be approached through very different aspects. In addition to the contribution presented in this subsection, the papers by Zelik (see Subsection 2.1) and by Zlotnik (see Subsection 2.6 below) are directly related to this topic.

[ChV] **Vladimir Chepyzhov and Mark Vishik**, *Attractors for nonautonomus Navier–Stokes system and other partial differential equations*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 135–266.

As was already mentioned, a description of attractors was a strong stimulus for mathematical research. Beginning with the 80's, the theory of *global attractors* was actively developed by many authors towards different directions, including the estimation of the Hausdorff dimension of attractors by basic scaling numbers (Reynolds, Grasshoff, etc.) of a flow. Attractors for nonautonomous equations were first studied by Chepyzhov and Vishik [**3**] who have made the main contribution to the field.

In the present paper, the authors treat the case of nonautonomous systems. The Hausdoff dimension of the global attractor can be infinite in the nonautonomous case, and, by this reason, the authors use the notion of an ε-entropy introduced by Kolmogorov for estimating the attractor size. Nonautonomous partial differential equations with oscillating external forces are analyzed. In particular, the authors consider the situation, where the amplitude of the oscillation grows infinitely, whereas the attractor remains bounded.

2.5. Statistical approach.

To derive an equation describing an instable movement, it is reasonable to replace unspecified forces by random forces with time-independent increments, instead of omitting unspecified forces altogether. Then one obtains a stochastic equation, i.e., a partial differential equation with white noise on the right-hand side. The presented results of Shirikyan lead to a very interesting setting of the problem that is adequate to described instable physical processes.

[Sh] **Armen Shirikyan**, *Exponential mixing for randomly forced partial differential equations. Method of coupling*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 155–188.

During many years, physicists were firmly convinced that the white noise possesses a smoothing effect on solutions to a partial differential equation. In the case of the complex Ginzburg–Landau equation, this conjecture finds its rigorous justification in the paper by Shirikyan presented in this collection. In fact, Shirikyan proves the ergodicity of stochastic partial differential equations, i.e., the uniqueness of the steady-state statistical solution even in the case, where the same partial differential equation, without white noise on the right-hand side, possesses many individual steady-state solutions belonging to an attractor of complicated structure. The smoothing action of the white noise is precisely to transform the set of individual steady-state solutions into a unique statistical steady-state solution. Using the coupling method, Shirikyan establishes a general criterion for the uniqueness of stationary measures and an exponential mixing property. The latter is understood as a certain kind of the Lyapunov exponential stability of the steady-state statistical solution. The method is then illustrated by the stochastic complex Ginzburg–Landau equation. Note that the results presented in [Sh] are based on an approach developed in a series of papers by Kuksin and Shirikyan (see references in [Sh]).

2.6. Water waves and free boundary problems.

The papers presented in this subsection are devoted to the study of delicate physical situations, where the surface separating a liquid and an external medium is not fixed. There are many different problems of such a kind. Some of them are discussed in our volumes.

[L] **David Lannes**, *Justifying asymptotics for 3D water-waves*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 1–22.

A motion of a perfect incompressible irrotational fluid under the influence of gravity is described by the free surface Euler (or water-wave) equations. These equations have rich structure, and many well-known equations in mathematical physics can be obtained as their asymptotic limits, for example, the Korteweg-de Vries equations, the Kadomtsev–Petviashvilii equations, the Boussinesq systems, the shallow water equations, the deep water models, etc. Lannes studies the validation of such asymptotics. Since the fluid is irrotational, it derives from a potential and therefore leads to the Dirichlet–Neumann operator on the free boundary. An asymptotic analysis of the Dirichlet–Neumann operator yields a linearized version of the problem. To reach the full nonlinear case, the perturbation method employing the Nash–Moser theorem is used.

[S] **Vsevolod Solonnikov**, *On problem of stability of equilibrium figures of uniformly rotating viscous incompressible liquid*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 189–254.

The free boundary problem governing the evolution of an isolated mass of a viscous incompressible fluid, subject to capillary and self-gravitation forces, is considered. The solvability of this problem in a finite time interval was established by the author in his previous publications. In the present paper, Solonnikov studies the stability of the solution corresponding to the rigid rotation of a liquid about the vertical axis with constant angular velocity. The main goal of this investigation is to show that the stability/instability is driven by the second variation of the energy functional, which has been done via analysis of the spectrum of the linearized operator in a neighborhood of the stationary regime. Then the perturbations are estimated in terms of the Hölder norms.

[Zl] **Alexander Zlotnik**, *On global in time properties of the symmetric compressible barotropic Navier–Stokes–Poisson flows in a vacuum*, In: Instability in Models Connected with Fluid Flows. II / Intern. Math. Ser. Vol. 7, Springer, 2008, pp. 329–376.

Unlike the papers [L] and [S] dealing with incompressible fluids (for instance, water) and several spatial dimensions, Zlotnik considers symmetric self-gravitating flows of a viscous compressible barotropic gas/fluid around a hard core with a free outer boundary in a vacuum. The density degenerates at the free boundary. Under spherical symmetry, the problem becomes one-dimensional relative to the spatial variables. Such problems arise in astrophysics. For large discontinuous initial data and general state functions (including increasing and not strictly increasing ones) the global-in-time bounds for solutions are established, which allows one to study of their large-time behavior. Results on the existence, nonexistence, and uniqueness of the corresponding static solutions are also presented.

2.7. Control theory.

Control theory gives the most natural point of view for engineering sciences. Indeed, instead of determining a solution in terms of data, one seeks to find the *most suitable data to produce the desired output.* This approach was first developed for time-dependent ordinary differential equations (see, for example, [**14**]). Due to the use of computers, advantages of functional analysis, and modern technology, this approach is now extended to distributed system. Note that control is closely related to the notion of *observability*, where frequencies of the solution play a crucial role. This fundamental fact was widely used by J.-L. Lions, one of the creators of control theory for PDEs. The main feature of this area is that many control problems arising in applications are ill posed in the sense of Hadamard.

[Sh] **Victor Isakov**, *Increased stability in the Cauchy problem for some elliptic equations*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 339–362.

Variations of the boundary data for elliptic equations generate fluctuations that show up everywhere in the domain. However, according to the regularizing properties of these problems, these fluctuations may be very small and the identification of their source is an ill-posed problem in the sense of Hadamard. It turns out that, in this setting, the most convenient tools for obtaining the best possible estimates are "Carleman estimates." Using these tools, Isakov derives some bounds which can be thought of as the increasing stability of the Cauchy problem for the Helmholtz equation with lower order terms when frequency is growing. These bounds hold under certain pseudoconvexity conditions on the surface for the Cauchy data and on the coefficient of the zero order term in the Helmholtz equation.

[AS] **Andrey Agrachev and Andrey Sarychev**, *Solid controllability in fluid dynamics*, In: Instability in Models Connected with Fluid Flows. I / Intern. Math. Ser. Vol. 6, Springer, 2008, pp. 1–36.

The authors consider the controllability and accessibility properties of the Navier–Stokes and Euler systems controlled by a low-dimensional force on the right hand side. After a survey of recent results, the authors establish new results for these systems on the two-dimensional sphere and generic two-dimensional Riemannian surfaces. They focus on geometric and Lie algebraic ideas, adopting the approach due to Arnold and Khesin [**1**] and making a connection with geometric methods in classical control theory. This paper should be especially interesting for those specialists, familiar with analytical methods, who wish to be introduced to the geometrical approach and to make a step towards more applied points of view.

3. Methods and Tools

To obtain the results presented in the volumes, the authors used well-known methods and their modifications or developed new approaches. Keeping in mind that mathematical methods are often as important as results they produce, we list the main methods and tools used by the contributors and indicate the corresponding references.

- Infinite dimensional geometric approach to fluid dynamics [AS]
- Nash–Moser theorem [L]
- Pseudodifferential calculus and harmonic analysis [L]
- Expansion of nonlinear part in terms of perturbation series [DS], [A]
- Nonlinear optic high frequency approximations [BF], [Ch]
- Poincaré–Sobolev operator [GMN]
- Resonant frequencies [GMN], [BF]
- White noise, stochastic methods, coupling method in particular [Sh]
- Hausdorff dimension, Kolmogorov entropy, attractors [ChV], [Ze].
- Carleman estimates [I]
- Weight Sobolev spaces [Ze]
- Moments of solutions (deterministic and random) [AS], [R], [Sh]
- Free boundary problems [S], [Zl]
- Conservation laws, hyperbolic systems with discontinuous coefficients [P]

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