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Lie Sphere Geometry

With Applications to Submanifolds

Second Edition

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To my sons,

Tom, Mark, and Michael

Preface to the First Edition

The purpose of this monograph is to provide an introduction to Lie's geometry of oriented spheres and its recent applications to the study of submanifolds of Euclidean space. Lie [104] introduced his sphere geometry in his dissertation, published as a paper in 1872, and used it in his study of contact transformations. The subject was actively pursued through the early part of the twentieth century, culminating with the publication in 1929 of the third volume of Blaschke's [10] *Vorlesungen über Differentialgeometrie*, which is devoted entirely to Lie sphere geometry and its subgeometries. After this, the subject fell out of favor until 1981, when Pinkall [146] used it as the principal tool in his classification of Dupin hypersurfaces in $\mathbf{R}^4$. Since that time, it has been employed by several geometers in the study of Dupin, isoparametric and taut submanifolds.

This book is not intended to replace Blaschke's work, which contains a wealth of material, particularly in dimensions two and three. Rather, it is meant to be a relatively brief introduction to the subject, which leads the reader to the frontiers of current research in this part of submanifold theory. Chapters 2 and 3 (chapter numbers from the second edition) are accessible to a beginning graduate student who has taken courses in linear and abstract algebra and projective geometry. Chapters 4 and 5 contain the applications to submanifold theory. These chapters require a first graduate course in differential geometry as a necessary background. A detailed description of the contents of the individual chapters is given in the introduction, which also serves as a survey of the field to this point in time.

I wish to acknowledge certain works which have been especially useful to me in writing this book. Much of Chapters 2 and 3 is based on Blaschke's book. The proof of the Cartan and Dieudonné theorem in Section 3.2 is taken from E. Artin's book [4], *Geometric Algebra*. Two sources are particularly influential in Chapters 4 and 5. The first is Pinkall's dissertation [146] and his subsequent paper [150], which have proven to be remarkably fruitful. Secondly, the approach to the study of Legendre submanifolds using the method of moving frames is due to Shiing-Shen Chern, and was presented in two papers by Chern and myself [37]–[38]. These two papers and indeed this monograph grew out of my work with Professor Chern during my 1985–

1986 sabbatical at Berkeley. I am very grateful to Professor Chern for many helpful discussions and insights.

I also want to thank several other mathematicians for their personal contributions. Katsumi Nomizu introduced me to Pinkall's work and Lie sphere geometry in 1982, and his seminar at Brown University has been the site of many enlightening discussions on the subject since that time. Thomas Banchoff introduced me to the cyclides of Dupin in the early seventies, when I was a graduate student, and he has provided me with several key insights over the years, particularly through his films. Patrick Ryan has contributed significantly to my understanding of this subject through many lectures and discussions. I also want to acknowledge helpful conversations and correspondence on various aspects of the subject with Steven Buyske, Sheila Carter, Leslie Coghlan, Josef Dorfmeister, Thomas Hawkins, Wu-Yi Hsiang, Nicolaas Kuiper, Martin Magid, Reiko Miyaoka, Ross Niebergall, Tetsuya Ozawa, Richard Palais, Ulrich Pinkall, Helmut Reckziegel, Chuu-Lian Terng, Gudlaugur Thorbergsson, and Alan West.

This book grew out of lectures given in the Brown University Differential Geometry Seminar in 1982–83 and subsequent lectures given to the Clavius Group during the summers of 1985–1989 at the University of Notre Dame, the University of California at Berkeley, Fairfield University and the Institute for Advanced Study. I want to thank my fellow members of the Clavius Group for their support of these lectures and many enlightening remarks. I also acknowledge with gratitude the hospitality of the institutions mentioned above.

I wish to thank my colleagues in the Department of Mathematics at the College of the Holy Cross, several of whom are my former teachers, for many insights and much encouragement over the years. I especially wish to mention my first teacher in linear algebra and real analysis, Leonard Sulski, who recently passed away after a courageous battle against leukemia. Professor Sulski was a superb, dedicated teacher, and a good and generous man. He will be missed by all who knew him.

While writing this book, I was supported by grants from the National Science Foundation (DMS-8907366 and DMS-9101961) and by a Faculty Fellowship from the College of the Holy Cross. This support is gratefully acknowledged.

I want to thank my three undergraduate research assistants from Holy Cross, Michele Intermont, Christopher Butler and Karen Purtell, who were also supported by the NSF. They worked through various versions of the manuscript and made many helpful comments. I also wish to thank the mathematics editorial department of Springer-Verlag for their timely professional help in preparing this manuscript for publication, and Kenneth Scott of Holy Cross for his assistance with the word-processing program.

Finally, I am most grateful to my wife, Patsy, and my sons, Tom, Mark, and Michael, for their patience, understanding and encouragement during this lengthy project.

August, 1991

Preface to the Second Edition

The most significant changes in the second edition are the following. First of all, this edition of the manuscript was prepared using the LaTeXdocument preparation system, and thus all of the numbers of the equations, theorems, etc., are different from the first edition.

I have added a new section Section 4.7 which describes the construction due to Ferus, Karcher, and Münzner [73] of isoparametric hypersurfaces with four principal curvatures, based on representations of Clifford algebras. Our treatment follows the original paper of Ferus, Karcher, and Münzner quite closely. I have also substantially revised the presentation of the invariance of tautness under Lie sphere transformations in Section 4.6, giving a different proof, due to J. C. Álvarez Paiva [2], who used functions whose level sets form a parabolic pencil of spheres rather than the usual distance functions to formulate tautness. This leads to a very natural proof of the Lie invariance of tautness.

Sections 5.2–5.4 regarding reducible Dupin hypersurfaces and the cyclides of Dupin have also been significantly revised, and 11 new figures illustrating the cyclides of Dupin have been added to Section 5.4. The introduction and several other places, for example Section 4.5, in the text where surveys of known results are given have all been updated to reflect the current state of research.

All 14 of the figures from the first edition were redone, and the second edition contains 14 additional figures. All of these figures were constructed by my colleague at Holy Cross, Andrew D. Hwang, using his ePiX program for constructing figures in the LaTeXpicture environment. I am most grateful to Professor Hwang for the excellent quality of the figures, and for his time and effort in constructing them. The project page for the ePiX program is: http://math.holycross.edu/∼ahwang/software/ePiX.html

In addition to the many mathematicians that I acknowledged in the preface to the first edition, I wish to thank Quo–Shin Chi and Gary Jensen, with whom I have collaborated on joint research over the past decade. This collaboration has been most stimulating and has led me to a deeper understanding of many aspects of this subject, in particular the method of moving frames and its applications to isoparametric and Dupin hypersurfaces.

I would also again like to thank the members of the Clavius Group for their encouragement and for their support of my lectures on Lie sphere geometry given at the University of Notre Dame in July, 2005 and the College of the Holy Cross in July, 2006. I also want to thank those institutions for their hospitality during the Clavius Group meetings.

While completing this second edition, I was supported by a grant from the National Science Foundation (DMS–0405529) and by a sabbatical leave from the College of the Holy Cross. This support is gratefully acknowledged. I also wish to thank my three recent undergraduate research assistants from Holy Cross, Ellen Gasparovic, Heather Johnson and Renee Laverdiere, who were also supported by my NSF grant, and who helped me in revising this book in various ways.

I am also grateful to Ann Kostant and her staff at Springer for their support and encouragement to complete a second edition of the book.

Finally, as with the first edition, I wish to thank my wife, Patsy, and my sons, Tom, Mark, and Michael, for their warm encouragement and support of all of my scholarly efforts, in particular, the second edition of this book.

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