Putnam and Beyond

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Life is good for only two things, discovering mathematics and teaching mathematics.
Siméon Poisson

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Preface

A problem book at the college level. A study guide for the Putnam competition. A bridge between high school problem solving and mathematical research. A friendly introduction to fundamental concepts and results. All these desires gave life to the pages that follow.

The William Lowell Putnam Mathematical Competition is the most prestigious mathematics competition at the undergraduate level in the world. Historically, this annual event began in 1938, following a suggestion of William Lowell Putnam, who realized the merits of an intellectual intercollegiate competition. Nowadays, over 2500 students from more than 300 colleges and universities in the United States and Canada take part in it. The name Putnam has become synonymous with excellence in undergraduate mathematics.

Using the Putnam competition as a symbol, we lay the foundations of higher mathematics from a unitary, problem-based perspective. As such, *Putnam and Beyond* is a journey through the world of college mathematics, providing a link between the stimulating problems of the high school years and the demanding problems of scientific investigation. It gives motivated students a chance to learn concepts and acquire strategies, hone their skills and test their knowledge, seek connections, and discover real world applications. Its ultimate goal is to build the appropriate background for graduate studies, whether in mathematics or applied sciences.

Our point of view is that in mathematics it is more important to understand *why* than to know *how*. Because of this we insist on proofs and reasoning. After all, mathematics means, as the Romanian mathematician Grigore Moisil once said, "correct reasoning." The ways of mathematical thinking are universal in today's science.

Putnam and Beyond targets primarily Putnam training sessions, problem-solving seminars, and math clubs at the college level, filling a gap in the undergraduate curriculum. But it does more than that. Written in the structured manner of a textbook, but with strong emphasis on problems and individual work, it covers what we think are the most important topics and techniques in undergraduate mathematics, brought together within the confines of a single book in order to strengthen one's belief in the unitary nature of mathematics. It is assumed that the reader possesses a moderate background, familiarity with the subject, and a certain level of sophistication, for what we cover reaches beyond the usual textbook, both in difficulty and in depth. When organizing the material, we were inspired by Georgia O'Keeffe's words: "Details are confusing. It is only by selection, by elimination, by emphasis that we get at the real meaning of things."

The book can be used to enhance the teaching of any undergraduate mathematics course, since it broadens the database of problems for courses in real analysis, linear algebra, trigonometry, analytical geometry, differential equations, number theory, combinatorics, and probability. Moreover, it can be used by graduate students and educators alike to expand their mathematical horizons, for many concepts of more advanced mathematics can be found here disguised in elementary language, such as the Gauss–Bonnet theorem, the linear propagation of errors in quantum mechanics, knot invariants, or the Heisenberg group. The way of thinking nurtured in this book opens the door for true scientific investigation.

As for the problems, they are in the spirit of mathematics competitions. Recall that the Putnam competition has two parts, each consisting of six problems, numbered A1 through A6, and B1 through B6. It is customary to list the problems in increasing order of difficulty, with A1 and B1 the easiest, and A6 and B6 the hardest. We keep the same ascending pattern but span a range from A0 to B7. This means that we start with some inviting problems below the difficulty of the test, then move forward into the depths of mathematics.

As sources of problems and ideas we used the Putnam exam itself, the International Competition in Mathematics for University Students, the International Mathematical Olympiad, national contests from the United States of America, Romania, Russia, China, India, Bulgaria, mathematics journals such as the *American Mathematical Monthly*, *Mathematics Magazine*, *Revista Matematică din Timişoara* (*Timişoara Mathematics Gazette*), *Gazeta Matematică* (*Mathematics Gazette, Bucharest*), *Kvant* (*Quantum*), *Kőzépiskolai Matematikai Lapok* (*Mathematical Magazine for High Schools* (*Budapest*)), and a very rich collection of Romanian publications. Many problems are original contributions of the authors. Whenever possible, we give the historical background and indicate the source and author of the problem. Some of our sources are hard to find; this is why we offer you their most beautiful problems. Other sources are widely circulated, and by selecting some of their most representative problems we bring them to your attention.

Here is a brief description of the contents of the book. The first chapter is introductory, giving an overview of methods widely used in proofs. The other five chapters reflect areas of mathematics: algebra, real analysis, geometry and trigonometry, number theory, combinatorics and probability. The emphasis is placed on the first two of these chapters, since they occupy the largest part of the undergraduate curriculum.

Within each chapter, problems are clustered by topic. We always offer a brief theoretical background illustrated by one or more detailed examples. Several problems are left for the reader to solve. And since our problems are true brainteasers, complete solutions are given in the second part of the book. Considerable care has been taken in selecting the most elegant solutions and writing them so as to stir imagination and stimulate research. We always "judged mathematical proofs," as Andrew Wiles once said, "by their beauty."

Putnam and Beyond is the fruit of work of the first author as coach of the University of Michigan and Texas Tech University Putnam teams and of the International Mathematical Olympiad teams of the United States and India, as well as the product of the vast experience of the second author as head coach of the United States International Mathematical Olympiad team, coach of the Romanian International Mathematical Olympiad team, director of the American Mathematics Competitions, and member of the Question Writing Committee of the William Lowell Putnam Mathematical Competition.

In conclusion, we would like to thank Elgin Johnston, Dorin Andrica, Chris Jeuell, Ioan Cucurezeanu, Marian Deaconescu, Gabriel Dospinescu, Ravi Vakil, Vinod Grover, V.V. Acharya, B.J. Venkatachala, C.R. Pranesachar, Bryant Heath, and the students of the International Mathematical Olympiad training programs of the United States and India for their suggestions and contributions. Most of all, we are deeply grateful to Richard Stong, David Kramer, and Paul Stanford for carefully reading the manuscript and considerably improving its quality. We would be delighted to receive further suggestions and corrections; these can be sent to rgelca@gmail.com.

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A Study Guide

The book has six chapters: Methods of Proof, Algebra, Real Analysis, Geometry and Trigonometry, Number Theory, Combinatorics and Probability, divided into subchapters such as Linear Algebra, Sequences and Series, Geometry, and Arithmetic. All subchapters are self-contained and independent of each other and can be studied in any order. In most cases they reflect standard undergraduate courses or fields of mathematics. The sections within each subchapter are best followed in the prescribed order.

If you are an *undergraduate student* trying to acquire skills or test your knowledge in a certain field, study first a regular textbook and make sure that you understand it very well. Then choose the appropriate chapter or subchapter of this book and proceed section by section. Read first the theoretical background and the examples from the introductory part; then do the problems. These are listed in increasing order of difficulty, but even the very first can be tricky. Don't get discouraged; put effort and imagination into each problem; and only if all else fails, look at the solution from the back of the book. But even if you are successful, read the solution, since many times it gives a new insight and, more important, opens the door toward more advanced mathematics.

Beware! The last few problems of each section can be very hard. It might be a good idea to skip them at the first encounter and return to them as you become more experienced.

If you are a *Putnam competitor*, then as you go on with the study of the book try your hand at the true Putnam problems (which have been published in three excellent volumes). Identify your weaknesses and insist on those chapters of *Putnam and Beyond*. Every once in a while, for a problem that you solved, write down the solution in detail, then compare it to the one given at the end of the book. It is very important that your solutions be correct, structured, convincing, and easy to follow.

An *instructor* can add some of the problems from the book to a regular course in order to stimulate and challenge the better students. Some of the theoretical subjects can also be incorporated in the course to give better insight and a new perspective. *Putnam* *and Beyond* can be used as a textbook for problem-solving courses, in which case we recommend beginning with the first chapter. Students should be encouraged to come up with their own original solutions.

If you are a *graduate student* in mathematics, it is important that you know and understand the contents of this book. First, mastering problems and learning how to write down arguments are essential matters for good performance in doctoral examinations. Second, most of the presented facts are building blocks of graduate courses; knowing them will make these courses natural and easy.

"Don't bother to just be better than your contemporaries or predecessors. Try to be better than yourself" (W. Faulkner).