Geometry of Quantum Theory

Second Edition

V.S. Varadarajan

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V.S. Varadarajan
Department of Mathematics
University of California
Los Angeles, CA 90024
USA

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TO MY PARENTS

PREFACE TO VOLUME I OF THE FIRST EDITION

The present work is the first volume of a substantially enlarged version of the mimeographed notes of a course of lectures first given by me in the Indian Statistical Institute, Calcutta, India, during 1964-65. When it was suggested that these lectures be developed into a book, I readily agreed and took the opportunity to extend the scope of the material covered.

No background in physics is in principle necessary for understanding the essential ideas in this work. However, a high degree of mathematical maturity is certainly indispensable. It is safe to say that I aim at an audience composed of professional mathematicians, advanced graduate students, and, hopefully, the rapidly increasing group of mathematical physicists who are attracted to fundamental mathematical questions.

Over the years, the mathematics of quantum theory has become more abstract and, consequently, simpler. Hilbert spaces have been used from the very beginning and, after Weyl and Wigner, group representations have come in conclusively. Recent discoveries seem to indicate that the role of group representations is destined for further expansion, not to speak of the impact of the theory of several complex variables and function-space analysis. But all of this pertains to the world of interacting subatomic particles; the more modest view of the microscopic world presented in this book requires somewhat less. The reader with a knowledge of abstract integration, Hilbert space theory, and topological groups will find the going easy.

Part of the work which went into the writing of this book was supported by the National Science Foundation Grant No. GP-5224. I have profited greatly from conversations with many friends and colleagues at various institutions. To all of them, especially to R. Arens, R. J. Blattner, R. Ranga Rao, K. R. Parthasarathy, and S. R. S. Varadhan, my sincere thanks. I want to record my deep thanks to my colleague Don Babbitt who read through the manuscript carefully, discovered many mistakes, and was responsible for significant improvement of the manuscript. My apologies are due to all those whose work has been ignored or, possibly, incorrectly (and/or insufficiently) discussed. Finally, I want to acknowledge that this book might never

have made its way into print but for my wife. She typed the entire manuscript, encouraged me when my enthusiasm went down, and made me understand some of the meaning of our ancient words,

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन । .*

To her my deep gratitude.

Spring, 1968 V. S. VARADARAJAN

* *Bhagavadgita, 2:47a.*

PREFACE TO THE SECOND EDITION

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन । .*

It was about four years ago that Springer-Verlag suggested that a revised edition in a single volume of my two-volume work may be worthwhile. I agreed enthusiastically but the project was delayed for many reasons, one of the most important of which was that I did not have at that time any clear idea as to how the revision was to be carried out. Eventually I decided to leave intact most of the original material, but make the current edition a little more up-to-date by adding, in the form of notes to the individual chapters, some recent references and occasional brief discussions of topics not treated in the original text. The only substantive change from the earlier work is in the treatment of projective geometry; Chapters II through V of the original Volume I have been condensed and streamlined into a single Chapter II. I wish to express my deep gratitude to Donald Babbitt for his generous advice that helped me in organizing this revision, and to Springer-Verlag for their patience and understanding that went beyond what one has a right to expect from a publisher.

I suppose an author's feelings are always mixed when one of his books that is comparatively old is brought out once again. The progress of Science in our time is so explosive that a discovery is hardly made before it becomes obsolete; and yet, precisely because of this, it is essential to keep in sight the origins of things that are taken for granted, if only to lend some perspective to what we are trying to achieve. All I can say is that there are times when one should look back as well as forward, and that the ancient lines, part of which are quoted above still capture the spirit of my thoughts.

Pacific Palisades,
Dec. 22, 1984

V. S. VARADARAJAN

* *Bhagavadgita, 2:47a.*

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INTRODUCTION

As laid down by Dirac in his great classic [1], the principle of superposition of states is the fundamental concept on which the quantum theory of atomic systems is to be erected. Dirac's development of quantum mechanics on an axiomatic basis is undoubtedly in keeping with the greatest traditions of the physical sciences. The scope and power of this principle can be recognized at once if one recalls that it survived virtually unmodified throughout the subsequent transition to a relativistic view of the atomic world. It must be pointed out, however, that the precise mathematical nature of the superposition principle was only implicit in the discussions of Dirac; we are indebted to John von Neumann for explicit formulation. In his characteristic way, he discovered that the set of experimental statements of a quantum mechanical system formed a projective geometry—the projective geometry of subspaces of a complex, separable, infinite dimensional Hilbert space. With this as a point of departure, he carried out a mathematical analysis of the axiomatic foundations of quantum mechanics which must certainly rank among his greatest achievements [1] [3] [4] [5] [6].

Once the geometric point of view is accepted, impressive consequences follow. The automorphisms of the geometry describe the dynamical and kinematical structure of quantum mechanical systems, thus leading to the *linear* character of quantum mechanics. The covariance of the physical laws under appropriate space-time groups consequently expresses itself in the form of projective unitary representations of these groups. The economy of thought as well as the unification of method that this point of view brings forth is truly immense; the Schrödinger equation, for example, is obtained from a representation of the time-translation group, the Dirac equation from a representation of the inhomogeneous Lorentz group. This development is the work of many mathematicians and physicists. However, insofar as the mathematical theory is concerned, no contribution is more outstanding than that of Eugene P. Wigner. Beginning with his famous article on time inversion and throughout his great papers on relativistic invariance [1] [3] [4] [5] [6], we find a beautiful and coherent approach to the mathematical description of the quantum mechanical world which achieves nothing less than the fusion of group theory and quantum mechanics, and moreover does this without compromising in any manner the axiomatic principles formulated by Dirac and von Neumann.

My own interest in the mathematical foundations of quantum mechanics received a great stimulus from the inspiring lectures given by Professor George W. Mackey at the University of Washington in Seattle during the summer of 1961. The present volumes are in great part the result of my interest in a detailed elaboration of the main features of the theory sketched by Mackey in those lectures. In sum, my indebtedness to Professor Mackey's lectures and to the books and papers of von Neumann and Wigner is immense and carries through this entire work.

There exist today many expositions of the basic principles of quantum mechanics. At the most sophisticated mathematical level, there are the books of von Neumann [1], Hermann Weyl [1] and Mackey [1]. But, insofar as I am aware, there is no account of the technical features of the geometry and group theory of quantum mechanical systems that is both reasonably self-contained and comprehensive enough to be able to include Lorentz invariance. Moreover, recent re-examinations of the fundamental ideas by numerous mathematicians have produced insights that have substantially added to our understanding of quantum foundations. From among these I want to single out for special mention Gleason's proof that quantum mechanical states are represented by the so-called density matrices, Mackey's extensive work on systems of imprimitivity and group representations, and Bargmann's work on the cohomology of Lie groups, particularly of the physically interesting groups and their extensions. All of this has made possible a conceptually unified and technically cogent development of the theory of quantum mechanical systems from a completely geometric point of view. The present work is an attempt to present such an approach.

Our approach may be described by means of a brief outline of the contents of the three parts that make up this work. The first part begins by introducing the viewpoint of von Neumann according to which every physical system has in its background a certain orthocomplemented lattice whose elements may be identified with the experimentally verifiable propositions about the system. For classical systems this lattice (called the logic of the system) is a Boolean σ-algebra while for quantum systems it is highly nondistributive. This points to the relevance of the theory of complemented lattices to the axiomatic foundations of quantum mechanics. In the presence of modularity and finiteness of rank, these lattices decompose into a direct sum of irreducible ones, called geometries. A typical example of a geometry is the lattice of subspaces of a finite dimensional vector space over a division ring. The theory of these vector geometries is taken up in Chapter II. The isomorphisms of such a geometry are induced in a natural fashion by semilinear transformations. Orthocomplementations are induced by definite semi-bilinear forms which are symmetric with respect to suitable involutive anti-automorphisms of the basic division ring. If the division ring is the reals, complexes or quaternions, this leads to the Hilbert space structures. In this chapter, we also examine the relation between axiomatic geometry and analytic geometry along classical lines with suitable modifications in order to handle the infinite dimensional case also. The main result of this chapter is the theorem which asserts that an abstractly given generalized geometry (i.e., one whose dimension need not be finite) of rank ≥ 4 is isomorphic to the lattice of all finite dimensional subspaces of a vector space over a division ring. The division ring is an invariant of the lattice.

The second part analyzes the structure of the logics of quantum mechanical systems. In Chapter III, we introduce the notion of an abstract logic (= orthocomplemented weakly modular σ-lattice) and the observables and states associated with it. It is possible that certain observables need not be simultaneously observable. It is proved that for a given family of observables to be simultaneously measurable, it is necessary and sufficient that the observables of the family be classically related, i.e., that there exists a Boolean sub σ-algebra of the logic in question to which all the members of the given family are associated. Given an observable and a state, it is shown how to compute the probability distribution of the observable in that state. In Chapter IV, we take up the problem of singling out the logic of all subspaces of a Hilbert space by a set of neat axioms. Using the results of Chapter II, it is proved that the standard logics are precisely the projective ones. The analysis of the notions of an observable and a state carried out in Chapter III now leads to the correspondence between observables and self-adjoint operators, and between the pure states and the rays of the underlying Hilbert space. The automorphisms of the standard logics are shown to be induced by the unitary and antiunitary operators. With this the von Neumann program of a deductive description of the principles of quantum mechanics is completed. The remarkable fact that there is a Hilbert space whose self-adjoint operators represent the observables and whose rays describe the (pure) states is thus finally established to be a consequence of the projective nature of the underlying logic.

The third and final part of the work deals with specialized questions. The main problem is that of a covariant description of a quantum mechanical system, the covariance being with respect to suitable symmetry groups of the system. The theory of such systems leads to sophisticated problems of harmonic analysis on locally compact groups. Chapters V, VI, and VII are devoted to these purely mathematical questions. The results obtained are then applied to yield the basic physical results in Chapters VIII and IX. In Chapter VIII, the Schrödinger equation is obtained and the relations between the Heisenberg and Schrödinger formulations of quantum mechanics are analyzed. The usual expressions for the position, momentum, and energy observables of a quantum mechanical particle are shown to be inevitable consequences of the basic axioms and the requirement of covariance. In addition, a classification of single particle systems is obtained in terms of the spin of the particle. The spin of a particle, which is so characteristic of quantum mechanics, is a manifestation of the *geometry* of the configuration space of the particle.

The final chapter discusses the description of free particles from the relativistic viewpoint. The results of Chapters V, VI and VII are used to obtain a classification of these particles in terms of their mass and spin. With each particle it is possible to associate a vector bundle whose square integrable sections constitute the Hilbert space of the particle. These abstract results lead to the standard transformation formulae for the (one particle) states under the elements of the relativity group. By taking Fourier transforms, it is possible to associate with each particle a definite wave equation. In particular, the Dirac equation of the free electron is obtained in this manner. The same methods lead to the localization in space, for a given time instant, of the particles of nonzero rest mass. The chapter ends with an analysis of Galilean relativity. It is shown that the free particles which are governed by Galilei's principle of relativity are none other than the Schrödinger particles of *positive* mass and arbitrary spin.

With this the program of obtaining a geometric view of the quantum mechanical world is completed. It is my belief that no other approach leads so clearly and smoothly to the fundamental results. It may be hoped that such methods may also lead to a successful description of the world of interacting particles and their fields. The realization of such hopes seems to be a matter for the future.

V. S. VARADARAJAN