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(continued after index)

Christopher G. Small

Functional Equations and How to Solve Them

Springer

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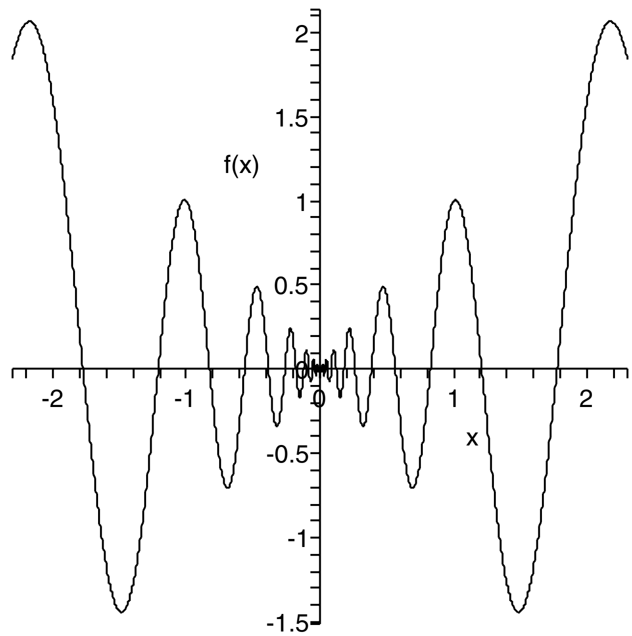
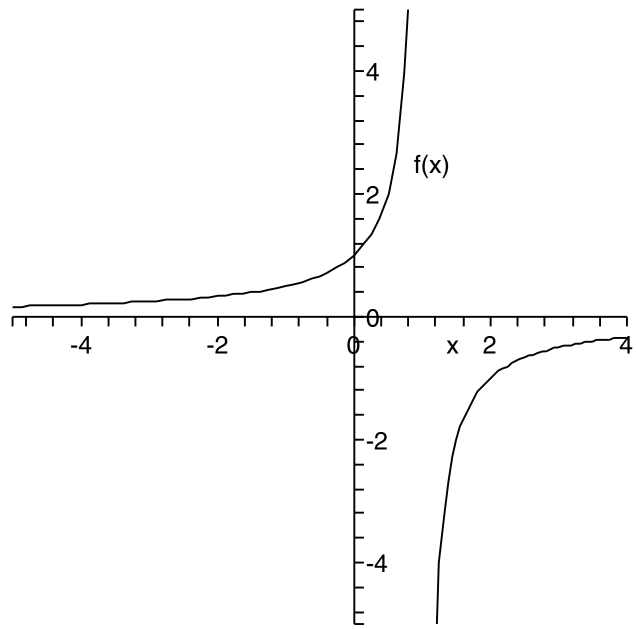


$$f(x) + f(2\,x) + f(3\,x) = 0$$

for all real x.

This functional equation is satisfied by the function $f(x) \equiv 0$, and also by the strange example graphed above. To find out more about this function, see Chapter 3.



$$f(f(f(x))) = x$$

Can you discover a function $f(x)$ which satisfies this functional equation?

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Preface

Over the years, a number of books have been written on the theory of functional equations. However, few books have been published on solving functional equations which arise in mathematics competitions and mathematical problem solving. The intention of this book is to go some distance towards filling this gap.

This work began life some years ago as a set of training notes for mathematics competitions such as the William Lowell Putnam Competition for undergraduate university students, and the International Mathematical Olympiad for high school students. As part of the training for these competitions, I tried to put together some systematic material on functional equations, which have formed a part of the International Mathematical Olympiad and a small component of the Putnam Competition. As I became more involved in coaching students for the Putnam and the International Mathematical Olympiad, I started to understand why there is not much training material available in systematic form. Many would argue that there is no theory attached to functional equations that are encountered in mathematics competitions. Each such equation requires different techniques to solve it. Functional equations are often the most difficult problems to be found on mathematics competitions because they require a minimal amount of background theory and a maximal amount of ingenuity. The great advantage of a problem involving functional equations is that you can construct problems that students at all levels can understand and play with. The great disadvantage is that, for many problems, few students can make much progress in finding solutions even if the required techniques are essentially elementary in nature. It is perhaps this view of functional equations which explains why most problem-solving texts have little systematic material on the subject. Problem books in mathematics usually include some functional equations in their chapters on algebra. But by including functional equations among the problems on polynomials or inequalities the essential character of the methodology is often lost.

As my training notes grew, so grew my conviction that we often do not do full justice to the role of theory in the solution of functional equations. The result of my growing awareness of the interplay between theory and problem application is the book you have before you. It is based upon my belief that a firm understanding of the theory is useful in practical problem solving with such equations. At times in this book, the marriage of theory and practice is not seamless as there are theoretical ideas whose practical utility is limited. However, they are an essential part of the subject that could not be omitted. Moreover, today's theoretical idea may be the inspiration for tomorrow's competition problem as the best problems often arise from pure research. We shall have to wait and see.

The student who encounters a functional equation on a mathematics contest will need to investigate solutions to the equation by finding all solutions (if any) or by showing that all solutions have a particular property. Our emphasis is on the development of those tools which are most useful in giving a family of solutions to each functional equation in explicit form.

At the end of each chapter, readers will find a list of problems associated with the material in that chapter. The problems vary greatly in difficulty, with the easiest problems being accessible to any high school student who has read the chapter carefully. It is my hope that the most difficult problems are a reasonable challenge to advanced students studying for the International Mathematical Olympiad at the high school level or the William Lowell Putnam Competition for university undergraduates. I have placed stars next to those problems which I consider to be the harder ones. However, I recognise that determining the level of difficulty of a problem is somewhat subjective. What one person finds difficult, another may find easy.

In writing these training notes, I have had to make a choice as to the generality of the topics covered. The modern theory of functional equations can occur in a very abstract setting that is quite inappropriate for the readership I have in mind. However, the abstraction of some parts of the modern theory reflects the fact that functional equations can occur in diverse settings: functions on the natural numbers, the integers, the reals, or the complex numbers can all be studied within the subject area of functional equations. Most of the time, the functions I have in mind are real-valued functions of a single real variable. However, I have tried not to be too restrictive in this. The reader will also find functions with complex arguments and functions defined on natural numbers in these pages. In some cases, equations for functions between circles will also crop up. Nor are functional inequalities ignored.

One word of warning is in order. You cannot study functional equations without making some use of the properties of limits and continuous functions. The fact is that many problems involving functional equations depend upon an assumption of such as continuity or some other regularity assumption that would usually not be encountered until university. This presents a difficulty for high school mathematics contests where the properties of limits and continuity cannot be assumed. One way to get around this problem is to substitute another regularity condition that is more acceptable for high school mathematics. Thus a problem where a continuity condition is natural may well get by with the assumption of monotonicity. Although continuity and monotonicity are logically independent properties (in the sense that neither implies the other) the imposition of a monotonicity condition in a functional equations problem may serve the same purpose as continuity. Another way around the problem is to ask students to provide a weaker conclusion that is not "finished" by invoking continuity. Asking students to determine the nature of a function on the rational numbers is an example of this. Neither solution to this problem is completely satisfactory. Fortunately, there are enough problems which can be posed and solved using high school mathematics to serve the purpose. More advanced contests such as the William Lowell Putnam Competition have no such restrictions in imposing continuity or convexity, and expect the student to treat these assumptions with mathematical maturity.

Some readers may be surprised to find that the chapter on functional equations in a single variable follows that on functional equations in two or more variables. However this is the correct order. An equation in two or more variables is formally equivalent to a family of simultaneous equations in one variable. So equations in two variables give you more to play with. I have had to be very selective in choosing topics in the third chapter, because much of the academic literature is devoted to establishing uniqueness theorems for solutions within particular families of functions: functions that are convex or real analytic, functions which obey certain order conditions, and so on. It would be easy to simply ignore the entire subject if it were not for the fact that functional equations in a single variable are commonplace in mathematics competitions. So I have done my best to present those tools and unifying concepts which occur periodically in such problems in both high school and university competitions. Chapter 3 has been written with a confidence that advanced high school students will adapt well to the challenge of a bit of introductory university level mathematics. The chapters can be read more-or-less independently of each other. There are some results in later chapters which depend upon earlier chapters. However, the reader who wishes to sample the book in random order can probably piece together the necessary information. The fact that it is possible to write a book whose chapters are not heavily dependent is a consequence of the character of functional equations. Unlike some branches of mathematics, the subject is wide, providing easier access from a number of perspectives. This makes it an excellent area for competition problems. Even tough functional equations are relatively easy to state and provide lots of "play value" for students who may not be able to solve them completely.

Because this is a book about problem solving, the reader may be surprised to find that it begins with a chapter of the history of the subject. It is my belief that the present way of teaching mathematics to students puts much emphasis on the tools of mathematics, and not enough on the intellectual climate which gave rise to these ideas. Functional equations were posed and solved for reasons that had much to do with the intellectual challenges of their times. This book attempts to provide a small glimpse of some of those reasons.

I have learned much about functional equations from other people. This book also owes much to others. So this preface would not be complete without some mention of the debts that I owe. I have learned much from the work of Janos Aczél, Distinguished Professor Emeritus at the University of Waterloo. The impact of his work and that of his colleagues is to be found throughout the following pages in places too numerous to mention. The initial stages of this monograph were written at the instigation of Pat Stewart and Richard Nowakowski. Sadly, Pat Stewart is no longer with us, and is missed by the mathematical community. Thank you, Patrick and Richard. Finally, I would like to thank Professor Ed Barbeau, who generously sent some of his correspondence problems to me. His encouragement and assistance are much appreciated.