

Part II

Geometry

If you have not yet read the preface, then please do so now.

Since you have read the preface, you already know that central to much of what we shall be looking at in Part III is the geometry of *excursion sets*:

$$A_u \equiv A_u(f, T) \triangleq \{t \in T : f(t) \geq u\} \equiv f^{-1}([u, \infty))$$

for random fields f over general parameter spaces T .

In order to do this, we are going to need quite a bit of geometry. In fact, we are going to need a lot more than one might expect, since the answers to the relatively simple questions that we shall be asking end up involving concepts that do not, at first sight, seem to have much to do with the original questions.

In Part III we shall see that an extremely convenient description of the geometric properties of A is its *Euler*, or *Euler–Poincaré characteristic*. In many cases, the Euler characteristic is determined by the fact that it is the unique integer-valued functional φ , defined on collections of nice A , satisfying

$$\varphi(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is ball-like,} \end{cases}$$

where by “ball-like” we mean homotopically equivalent²² to the unit N -ball, $B^N = \{t \in \mathbb{R}^N : |t| \leq 1\}$, and with the additivity property that

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B).$$

More global descriptions follow from this. For example, if A is a nice set in \mathbb{R}^2 , then $\varphi(A)$ is simply the number of connected components of A minus the number of holes in it. In \mathbb{R}^3 , $\varphi(A)$ is given by the number of connected components, minus the number of “handles,” plus the number of holes. Thus, for example, the Euler characteristics of a baseball, a tennis ball, and a coffee cup are, respectively, 1, 2, and 0.

One of the basic properties of the Euler characteristic is that it is determined by the homology class of a set. That is, smooth transformations that do not change the basic “shape” of a set do not change its Euler characteristic. In Euclidean spaces finer geometric information, which *will* change under such transformations, lies in the so-called *Minkowski functionals*. However, even if we were to decide to do without this finer information, we would still arrive at a scenario involving it. We shall see, again in Part III, that the expected value of $\varphi(A_u(f, T))$ will, in general, involve the Minkowski functionals of T . This is what was meant above about simple questions leading to complicated answers.

There are basically two different approaches to developing the geometry that we shall require. The first works well for sets in \mathbb{R}^N that are made up of the finite union of simple building blocks, such as convex sets. For many of our readers, we imagine that this will suffice. The basic framework here is that of integral geometry.

The second, more fundamental approach is via the differential geometry of abstract manifolds. As described in the preface, this more general setting has very concrete applications, and moreover, often provides more powerful and elegant proofs

²² The notion of homotopic equivalence is formalized below by Definition 6.1.4. However, for the moment, “ball-like” will be just as useful a concept.

for a number of problems related to random fields even on the “flat” manifold \mathbb{R}^N . This approach is crucial if you want to understand the full theory. Furthermore, since some of the proofs of later results, even in the integral-geometric setting, are more natural in the manifold scenario, it is essential if you need to see full proofs. However, the jump in level of mathematical sophistication between the two approaches is significant, so that unless you feel very much at home in the world of manifolds you are best advised to read the integral-geometric story first.

Chapter 6 handles all the integral geometry that we shall need. The treatment there is detailed, complete, and fully self-contained. This is not the case when we turn to differential geometry, where the theory is too rich to treat in full. We start with a quick and nasty revision of basic differential geometry in Chapter 7, most of which is standard graduate-course material. Chapter 8 treats piecewise smooth manifolds, which provide the link between the geometry of Chapter 6, with its sharp corners and edges, and the smooth manifolds of Chapter 7. In Chapter 9 we look at Morse theory in the piecewise smooth setting. While Morse theory in the smooth setting is, once again, standard material, the piecewise smooth case is less widely known. In fact, Chapter 9 actually contains some new results that, unlike most of the rest of Part II, we were not able to find elsewhere.

In the passage from integral to differential geometry a number of things will happen. Among them, the space T (time, or multidimensional time) will become M (manifold)²³ and the Minkowski functionals will become the *Lipschitz–Killing curvatures*.

While Lipschitz–Killing curvatures are well-known objects in global differential geometry, we do not imagine that they will be too well known to the probabilist reader. Hence we devote Chapter 10 to a study of the so-called *tube formulas*, originally due to Hermann Weyl [168] in Euclidean spaces. In addition, as we have already noted in the preface, some of the proofs are different from what is found in the standard geometry literature, in that they rely on properties of Gaussian distributions. Furthermore, Chapter 10 also discusses a relatively new generalization of the Weyl results to manifolds in Gauss space, due to JET [158].

The tube formulas of Chapter 10 can also be exploited to develop a formal approximation to the excursion probability

$$\mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\}$$

for certain Gaussian fields with finite orthonormal expansions,²⁴ and we shall look at this in Sections 10.2 and 10.6.

All of Part II, old and new, is crucial for understanding the general proofs of Part III.

²³ A true transition from T to M would also have the elements t of T becoming elements p (points) of M . However, as we have already mentioned, this seems to be too heavy a psychological price for a probabilist to pay, so we shall remain with points $t \in M$. For this mortal sin of misnotation we beg forgiveness from our geometer colleagues.

²⁴ The most general case will be treated in detail in Chapter 14 via Morse-theoretic techniques.