Part I

Gaussian Processes

If you have not yet read the preface, then please do so now.

Since you have read the preface, you already know a number of important things about this book, including the fact that Part I is about Gaussian random fields.

The centrality of Gaussian fields to this book is due to two basic factors:

- Gaussian processes have a rich, detailed, and very well-understood general theory, which makes them beloved by theoreticians.
- In applications of random field theory, as in applications of almost any theory, it is important to have specific, explicit formulas that allow one to predict, to compare theory with experiment, etc. As we shall see in Part III, it will be only for Gaussian (and related; cf. Section 1.4.6 and Chapter 15) fields that it is possible to derive such formulas, and then only in the setting of excursion sets.

The main reason behind both these facts is the convenient analytic form of the multivariate Gaussian density, and the related properties of Gaussian fields. This is what Part I is about.

There are five main collections of basic results that will be of interest to us. Rather interestingly, although later in the book we shall be interested in Gaussian fields defined over various types of manifolds, the basic theory of Gaussian fields is actually independent of the specific geometric structure of the parameter space. Indeed, after decades of polishing, even proofs gain little in the way of simplification by restricting to special cases even as simple as $\mathbb{R}$. Thus, at least for a while, we can and shall work in as wide as possible generality, working with fields defined only on topological spaces to which we shall assign a natural (pseudo)metric induced by the covariance structure of the field.

The first set of results that we require, along with related information, form Chapter 1 and are encapsulated in different forms in Theorems 1.3.3 and 1.3.5 and their corollaries. These give a sufficient condition, in terms of metric entropy, ensuring the sample path boundedness and continuity of a Gaussian field along with providing information about moduli of continuity. While this entropy condition is also necessary for *stationary* fields, this is not the case in general, and so for completeness we look briefly at the majorizing measure version of this theory in Section 1.5. However, it will be a rare reader of *this* book who will ever need the more general theory.

To put the seemingly abstract entropy conditions into focus, these results will be followed by a section with a goodly number of extremely varied examples. Despite the fact that these cover only the tip of a very large iceberg, their diversity shows the power of the abstract approach, in that all can be treated via the general theory without further probabilistic arguments. The reader who is not interested in the general Gaussian theory, and cares mainly about the geometry of fields on $\mathbb{R}^N$ or on smooth manifolds, need only read Sections 1.4.1 and 1.4.2 on continuity and differentiability in this scenario, along with the early parts of Section 1.4.3, needed for understanding the spectral representation of Chapter 5.

Chapter 2 contains the Borell–TIS (Borell–Tsirelson–Ibragimov–Sudakov) inequality and Slepian inequalities (along with some of their relatives). The Borell–TIS inequality gives a universal bound for the *excursion probability*

$$\mathbb{P}\left\{ \sup_{t \in T} f(t) \geq u \right\}, \tag{0.0.2}$$

$u > 0$, for *any* centered, continuous Gaussian field. As such, it is a truly basic tool of Gaussian processes, somewhat akin to Chebyshev's inequality in statistics or maximal inequalities in martingale theory. Slepian's inequality and its relatives are just as important and basic, and allow one to use relationships between covariance functions of Gaussian fields to compare excursion probabilities and expectations of suprema.

The main result of Chapter 3 is Theorem 3.1.1, which gives an expansion for a Gaussian field in terms of deterministic eigenfunctions with independent $N(0, 1)$ coefficients. A special case of this expansion is the Karhunen–Loève expansion of Section 3.2, with which we expect many readers will already be familiar. Together with the spectral representations of Chapter 5, they make up what are probably the most important tools in the Gaussian modeler's bag of tricks. However, these expansions are also an extremely important theoretical tool, whose development has far-reaching consequences.

Chapter 4 serves as a basic introduction to what is also one of the central topics of Part III, the computation of the excursion probabilities for a zero-mean Gaussian field and a general parameter space T. In Part III we shall develop highly detailed expansions of the form

$$\mathbb{P}\left\{ \sup_{t \in T} f(t) \geq u \right\} = u^{\alpha} e^{-u^2/2\sigma_T^2} \sum_{j=0}^{n} C_j u^{-j} + \text{error},$$

for large u, appropriate parameters α, σ_T^2, n, and C_j that depend on both f and T. However, to do this, we shall have to place specific assumptions on the parameter space T, in particular assuming that it is a piecewise smooth manifold.[6] Without these assumptions, the best that one can do is to identify α, σ_T^2, and occasionally C_0, and this is what Chapter 4 will do. Furthermore, to keep the treatment down to a reasonable length, we shall generally concentrate only on upper bounds, rather than expansions, for Gaussian excursion probabilities.[7]

Chapter 5, the last of Part I, is somewhat different from the others in that it is not really about Gaussian processes, but about stationarity and isotropy in general. The main reason for the generality is that limiting oneself to the Gaussian scenario gains us so little that it is not worthwhile doing so. The results here, however, will be crucial for many of the detailed calculations of Part III.

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[6] For those of you who are already comfortable with the theory of stratified manifolds, our "piecewise smooth manifolds" are Whitney stratified manifolds with convex support cones.

[7] There is also a well-developed theory of a Poisson limit nature for probabilities of the form $\mathbb{P}\{\sup_{t \in T_u} f(t) \geq \eta(x, u)\}$, for which $\eta(x, u)$ and the size of T_u grow (to infinity) with u. In this case, one searches for growth rates that give a limit dependent on x, but not u. You can find more about this in Aldous [10], which places the Gaussian theory within a much wider framework of limit results, or in Leadbetter, Lindgren, and Rootzén [97] and Piterbarg [126], which give more detailed and more rigorous accounts for the Gaussian and Gaussian-related situation.

There are a number of ways in which you can read Part I of this book. You should definitely start with Sections 1.1 and 1.2, which have some boring, standard, but important technical material. From then on, it is very much up to you, since the remainder of Chapter 1 and the other four chapters of Part I are more or less independent of one another. A reviewer of the book suggested going from Section 1.2 directly to Chapter 3 to read about how to construct examples of Gaussian processes via orthogonal expansions, a suggestion that certainly makes historical sense and would also be more natural for an analyst rather than a probabilist. With or without Chapter 3, one can go from Section 1.2 to Section 1.3 to learn a little about entropy methods and from there directly to Chapter 4 to get quickly to extremal properties, one of the key topics of this book. Chapter 2, on the Borell and Slepian inequalities, can follow this. We obviously wrote the book in the order that seemed most logical to us, but you do have a fair number of choices in how to read Part I.

Finally, we repeat what was already said in the preface, that there is nothing new in Part I beyond perhaps the way some things are presented, and that as a treament of the basic theory of Gaussian fields it is *not* meant to be exhaustive. There are now many books covering various aspects of this theory, including those by Bogachev [28], Dudley [56], Fernique [67], Hida and Hitsuda [77], Janson [86], Ledoux and Talagrand [99], Lifshits [105], and Piterbarg [126]. In terms of what will be important to us, [56] and [99] stand out from the pack, perhaps augmented with Talagrand's review [154]. Finally, while not as exhaustive[8] as the others, you might find RJA's lecture notes [3], augmented with the corrections in Section 2.1 below, a user-friendly introduction to the subject.

[8] Nor, perhaps, as exhausting.