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Douglas S. Bridges and Luminiţa Simona Vîţă

Techniques of Constructive Analysis

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Ideal pierdut în noaptea unei lumi ce nu mai este,
Lume ce gândea în basme şi vorbea în poezii,
O! te văd, te-aud, te cuget, tânără şi dulce veste
Dintr-un cer cu alte stele, cu-alte raiuri, cu alţi zei.

—Mihai Eminescu, "Venere şi Madonă"

Oh, ideal lost in night-mists of a vanished universe:
People who would think in legends—all a world who spoke in verse;
I can see and think and hear you—youthful scout which gently nods
From a sky with different starlights, other Edens, other gods.

—Mihai Eminescu, "Venus and Madonna"
(translated by Andrei Bantaş)

"My homework does exist but no one can find it"

This image is a courtesy of The Times of London. Printed in the February 3, 2004 issue.

Preface

ROSENCRANTZ: *Shouldn't we be doing something... constructive?*

GUILDENSTERN: *What did you have in mind?*

—Tom Stoppard, *Rosencratz and Guildenstern Are Dead*

We have written this book in order to provide an introduction to constructive analysis, emphasising techniques and results that have been obtained in the last twenty years. The intended readership comprises senior undergraduates, postgraduates, and professional researchers in mathematics and theoretical computer science. We hope that our work will help spread the message that doing mathematics constructively is interesting (it can even be fun!) and challenging, and produces new, deep computational information.

An appreciation of the distinction between constructive and nonconstructive has become more widespread in this era of computers. Nevertheless, there are few books devoted to the development of mathematics in a rigorously constructive/computable fashion, although there are some, primarily concentrating on logic and foundations, in which the odd chapter deals with constructive mathematics proper as distinct from its underlying logic or set theory. It is now almost forty years since the publication of Errett Bishop's seminal monograph *Foundations of Constructive Analysis* [9], which in our view is one of the most remarkable intellectual documents of the twentieth century, and more than twenty since the appearance of its outgrowth [12]. In the intervening years there has been considerable activity in constructive analysis, algebra, and topology; in related foundational areas such as type theory [69]; and in the relation between constructive mathematics and computer science (for example, program extraction from proofs [42, 70, 51]). Believing that a new introduction to the mathematical, as distinct from the foundational, side of the subject is overdue, we embarked upon this monograph.

Our book is intended not to replace, but to supplement, Bishop's original classic [9] and the later volume [12] based thereon. Both of those two monographs cover aspects of analysis, such as Haar measure and commutative Banach algebras, that we do not mention. We cover some topics that are found in [9] and [12] (it would be almost inconceivable to produce a book like ours, dealing with constructive mathematics for nonexperts, without proving, for example, basic results about locatedness and total boundedness); but we have tried to provide improved proofs whenever possible. However, much of the material we present was simply not around at the time of writing of [9] or [12].

Instead of systematically developing analysis, beginning with the real line and continuing through metric, normed, and Hilbert spaces to its higher reaches, we have chosen to write the chapters around certain themes or techniques (hence our title). For example, Chapter 3 is devoted to the λ-technique, which, since its first use in the proof of Lemma 7 on page 177 of [9], has become a surprisingly powerful tool with applications in many areas of constructive analysis. A major influence in the application of the λ-technique was Ishihara's remarkable paper [60], which showed that a subtle use of the technique could enable us to prove disjunctions whose proof, although trivial with classical logic, appears at first sight to be constructively out of the question. This paper opened up many new pathways in constructive analysis.

Chapter 1 introduces constructive mathematics and lays the foundations for the later chapters. In Chapter 2 we first present a new construction of the real numbers, motivated by ideas in [2]. After deriving standard properties such as the completeness of $\mathbb{R}$, we introduce metric spaces, with the major theme of locatedness, and normed linear spaces. When we discuss metric, normed, and Hilbert spaces, we assume some familiarity with the standard classical definitions of those concepts and with those elementary classical properties that pass over unchanged to the constructive setting.

Chapter 3 we have already referred to. The main theme of Chapter 4 is finite-dimensionality, but the chapter concludes with an introduction to Hilbert spaces.

Chapter 5 deals with convexity in normed spaces. Starting with some elementary convex geometry in $\mathbb{R}^n$, the chapter goes on to handle separation and Hahn–Banach theorems, locally convex spaces, and duality. Following Bishop, we describe those linear functionals that are weak*-uniformly continuous on the unit ball of the dual space. We then give a new application of the technique used to prove that result, thereby characterising certain continuous linear functionals on the space of bounded operators on a Hilbert space.

In Chapter 6 we derive a range of results associated with the theme of locatedness and with the λ-technique introduced in Chapter 3. We pay particular attention to necessary and sufficient conditions for convex subsets of a normed space to be located, and to connections between properties of an operator on a Hilbert space and those of its adjoint—when that adjoint exists: it may not always do so constructively. The final section of the book deals with a relatively recent version of Baire's theorem and its applications, and culminates in constructive versions of three of the big guns in functional analysis: the open mapping, inverse mapping, and closed graph theorems.

Which parts of the book deal with new material, compared with what appeared in [12]? We have already mentioned the new construction of the real numbers, in Chapter 2. Notable novelties in the later chapters include all but one result in Chapter 3 on the λ-technique; the section on convexity, Ishihara's results on exact Hahn–Banach extensions, and our characterisation theorem for certain continuous linear functionals, all in Chapter 5; and virtually all of Chapter 6. Throughout the book there are what we hope will be seen as improvements and simplifications of proofs of many results that were given in [9] or [12].

What do we mean by "constructive analysis" in the title of this book? We do *not* mean analysis carried out with the usual "classical" logic within a framework, such as recursive function theory, designed to capture the concept of computability. In our view, such a notion of constructive has at least two drawbacks. First, by working within, say, the recursive setting, it can make the mathematics look less like normal mathematics and much harder to read. Secondly, the recursive constraint removes the possibility of other interpretations of the mathematics, such as Brouwer's intuitionistic one [48]. Our approach, on the other hand, has neither of these features: the mathematics looks and reads just like the mathematics one is used to from undergraduate days, and all our proofs and results are valid in several models. They are valid in the recursive model, in intuitionistic mathematics, and, we believe, in any of the models for "computable mathematics" (including Weihrauch's Type Two Effectivity Theory [91], within which Andrej Bauer has recently found a realisability interpretation of constructive mathematics within Weihrauch's theory [5]). *They are also valid proofs in standard mathematics with classical logic.* For example, our proof of the Hahn–Banach theorem (Theorem 5.3.3) is, as it stands, a valid algorithmic proof of the classical Hahn–Banach theorem. Moreover—and this is one advantage of a constructive proof in general—our proof embodies an algorithm for the construction of the functional whose existence is stated in the theorem. This algorithm can be extracted from the proof, and, as an undeserved bonus, the proof itself demonstrates that the algorithm is correct or, in computer science parlance, "meets its specifications".[1]

So how do we achieve all this? Simply by changing the logic with which we do our mathematics! Instead of using classical logic, we systematically use intuitionistic logic, which was abstracted by Heyting [52] from the practice of Brouwer's intuitionistic mathematics. The remarkable fact is that every proof carried out with intuitionistic logic is fully constructive/algorithmic. (Is this the "secret on the point of being blabbed" that appears in the epigraph to Bishop's book?) Unfortunately, too few mathematicians outside the mathematical logic community are aware of this serendipity and dismiss both intuitionistic logic and constructive mathematics as at best a marginal curiosity. This contrasts sharply with the theoretical computer science community, in which there is considerable knowledge of, and interest in, the computational power of intuitionistic logic.

[1]We do not carry out program-extraction from proofs in our book. For more on this topic see [42, 51, 70].

Reading constructive mathematics demands careful interpretation. A theorem in this book might look like a familiar one from classical analysis, but with more complicated hypotheses and proof. However, the statement of the theorem will be phrased so that the explicit algorithmic interpretation is left to the reader; and the additional hypotheses will be necessary for a constructive proof, which will contain algorithmic information that is excluded from the classical proof by the latter's use of principles outside intuitionistic logic. Consider, for example, the following statement:

> (*) *Let C be an open convex subset of a normed space X, let $\xi \in C$, and let $z \in X$ be bounded away from C. Then the boundary of C intersects the segment $[\xi, z]$ joining ξ and z.*

This is trivial to prove classically; but to find/construct the (necessarily unique) point in which the boundary of C intersects $[\xi, z]$ is a totally different matter. The constructive theorem (Proposition 5.1.5 below) requires us to postulate that the union of C and its metric complement $-C$ (the set of points bounded away from C) be dense in X, and that X itself be a complete normed space. The constructive proof, though elementary, requires some careful geometrical estimation that would be supererogatory in the natural classical proof by contradiction. The benefit of that estimation and of the use of intuitionistic logic is that we could extract from the constructive proof an implementable algorithm for finding the point where the segment crosses the boundary. In turn, this would enable us to produce an algorithm for constructing separating hyperplanes and Hahn–Banach extensions of linear functionals, under appropriate hypotheses.

We could have made the algorithmic interpretation of the constructive version of (*) explicit by stating the proposition in this way:

> *There is a "boundary crossing algorithm" that, applied to the data consisting of* (i) *an open convex set C in a Banach space X such that $C \cup -C$ is dense in X,* (ii) *a point ξ of C, and* (iii) *a point z of $-C$, constructs the point where the boundary of C intersects the segment $[\xi, z]$.*

Even this is not really explicit enough. A full description of the data to which the boundary crossing algorithm applies would require explicit information about the algorithms for such things as these: membership of C; the convergence of Cauchy sequences in X; the computation, for given x in X and $\varepsilon > 0$, of a point y of $C \cup -C$ such that $\|x - y\| < \varepsilon$ (and even the decision between the cases "$y \in C$" and "$y \in -C$"); and so on. Such explicit description of algorithmic hypotheses would become an ever greater burden on writer and reader alike as the book probed deeper and deeper into abstract analysis. It is a matter of sound sense, even sanity, to unburden ourselves from the outset, relying on the reader's native wit in the interpretation of the statements of our constructive lemmas, propositions, and theorems.

We should make it clear that we are not advocating the exclusive use of intuitionistic logic in mathematics. That logic is, we believe, the natural and right one to use when dealing with the constructive content of mathematics. To abandon classical logic in those fields (such as the higher reaches of set theory) where constructivity is of little or no significance makes no sense whatsoever. Nevertheless, it is remarkable how much mathematics actually has what Bishop called "a deep underpinning of constructive truth".

Christchurch, New Zealand
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Luminiţa Simona Vîţă

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